Forecasting

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Ground Rules

- Be polite.
  - listen while presenters are talking
  - ask questions in a polite manner
- Be punctual.
  - your instruction will be here from 6:30-8:30
  - so should you
  - some material will build on previous material: try to come weekly
- Be engaged.
  - listen while presenters are talking
  - don't check your phone
  - don't do your other homework
  - work on the problems/exercises given to you

Question

Why do we care so much about the weather?

Question

How do we predict the weather? Discuss with those around you ways you would predict the weather.

History

http://timemapper.okfnlabs.org/manunicast/history-of-weather-forecasting#0

Why is it so difficult to predict weather?

- To predict the future, our best guess is to use what we know about the past, even though there might not be a good reason for the past to influence the future.
- What factors might have an effect on the weather?

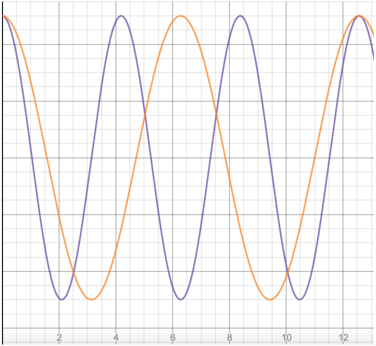
Factors Influencing Weather

- Geographical location
- Time of year/day (Solar distance)
- Water Cycle (Proximity to Water, Recent Precipitation)
- Wind
- Air pressure

How accurate do we need to be?

1 $y = 5 \cdot \cos(-1.5 \cdot x) + 10$

2 $y = 5 \cdot \cos(-x) + 10$



Markov Matrices

- How do we predict the weather?
- We will used a simplified model based on Markov matrices.

Example

Let's say we do a search on historical data and found the following information on rainy (R) vs sunny (S) days:

S	R	S	S	S	R	R	R	S	S

Calculations

We conclude that after a sunny day, $3/5$ times we have another sunny day and $2/5$ times we have a rainy day. After a rainy day, we have a $2/4$ chance of another rainy day and a $2/4$ chance of a sunny day.

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What is the probability that tomorrow is rainy? Answer: $3/5$

What about the probability that two days from now we will have a sunny day?

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What is the probability that tomorrow is rainy? Answer: $3/5$
What about the probability that two days from now we will have a sunny day? Answer: $(3/5)(3/5) + (2/5)(1/2) = 14/25$
What about the probability that 3 days from now it will be sunny?

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What about the probability that two days from now we will have a sunny day? Answer: $(3/5)(3/5) + (2/5)(1/2) = 14/25$

What about the probability that 3 days from now it will be sunny?Answer: $(14/25)(3/5) + (9/25)(1/2) = 129/250$

What about 100 days from now?

Framework

We need a more efficient way to do these computations. We can accomplish this goal using a mathematical object called a matrix.

Definition

An $n \times m$ **matrix** is an array of entries with n rows and m columns (for some positive integers m and n).

For example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a 2 by 2 matrix.

Operations on matrices

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Let's use this to multiply the two matrices $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

Visualization

https://www.youtube.com/watch?v=bFeM4ICRtOM

Practice some on your own!

Time for some practice!

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix} \cdot \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\-1 & 5\end{bmatrix}$$

$$\begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}^2 \qquad \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix}^3$$

$$\begin{bmatrix}3 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}2 & -5\\-1 & 3\end{bmatrix} \qquad \begin{bmatrix}1 & 2 & 0\\2 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\1 & 2 & 3\\1 & 1 & 1\end{bmatrix}$$

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It will also turn out that this definition corresponds precisely to the situation we need.

Revisit the Weather

Recall:

S	R	S	S	S	R	R	R	S	S

We conclude that after a sunny day, $3/5$ times we have another sunny day and $2/5$ times we have a rainy day. After a rainy day, we have a $2/4$ chance of another rainy day and a $2/4$ chance of a sunny day. In matrix form, this would correspond to:

$$\begin{array}{cc} & \begin{array}{cc} \text{S} & \text{R} \end{array} \\ \begin{array}{c} \text{S} \\ \text{R} \end{array} & \begin{bmatrix} 3/5 & 1/2 \\ 2/5 & 1/2 \end{bmatrix} \end{array}$$

Let's call this matrix M.

Repeat the previous computations

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Take the previous matrix and apply it to the current state. We know that today it is sunny which gives the matrix (vector) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
(That is, there is a 100% chance today that it is sunny and a 0% chance that it is rainy). Then

$$\begin{bmatrix} 3/5 & 1/2 \\ 2/5 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$$

So the probability that tomorrow it will be sunny is $3/5$ and a $2/5$ chance that tomorrow it will be rainy.

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So the probability that tomorrow it will be sunny is $3/5$ and a $2/5$ chance that tomorrow it will be rainy.
How do you think we would do two days? What about three days? What about 100 days?

Even still

To do this 100 days in advance, we would still need to do this operation 100 times which is very inefficient.

Go back to the first exercise. Which of the given matrix operations were easier to deal with?

Diagonal Matrix

Definition

A **diagonal matrix** is a square matrix with all zeroes in non-diagonal entries. For example,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 359 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2.457 & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

Changing Matrices

Can we take our matrix $M = \begin{bmatrix} 3/5 & 1/2 \\ 2/5 & 1/2 \end{bmatrix}$ and somehow change it into a diagonal matrix?

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Yes! ... sort of.

Mathemagic

Let $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/10 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 4/5 & -1 \end{bmatrix}$ and let $T = \begin{bmatrix} 5/9 & 5/9 \\ 4/9 & -5/9 \end{bmatrix}$.
Compute TS and SDT.

Spoiler

Therefore, since $M = SDT$, we see that $M^2 = (SDT)^2 = SDTSDT = SD^2T$ and similarly, $M^{100} = SD^{100}T$. Let's compute the last one. Therefore, the probability that it will be sunny 100 days form now is...

Other Forecasts

What other things might we want to attempt to forecast? How has progress been in those fields?

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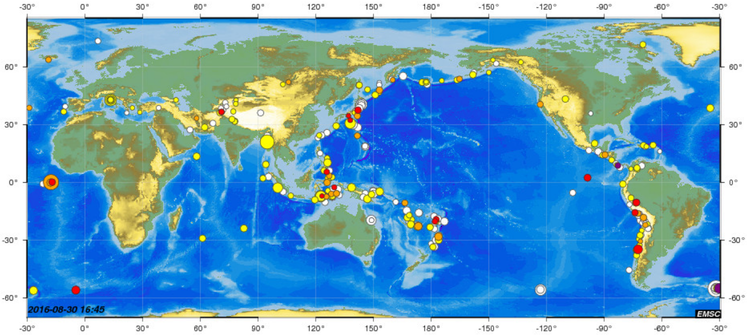
- Elections
- Earthquakes
- Stock Markets
- Airline Ticket Prices

Earthquakes - Two Week Span



http://quakes.globalincidentmap.com/

References

- http://earthobservatory.nasa.gov/Features/WxForecasting/wx2.php
- http://timemapper.okfnlabs.org/manunicast/history-of-weather-forecasting#0
- “The Signal and the Noise” - Nate Silver