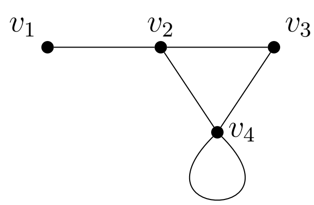
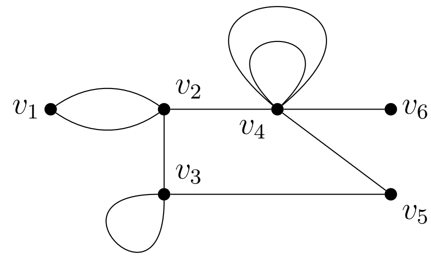
Intermediate Math Circles
Wednesday, February 15, 2017
Problem Set 2

1. Write down the adjacency matrix for each of the following graphs.

(a)



(b)



2. Draw a graph corresponding to each of the following adjacency matrices.

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

3. Explain how to decide whether or not a graph is simple by looking at its adjacency matrix alone. Decide which of the following adjacency matrices corresponds to a simple graph *without drawing the graph.*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

4. (a) Let G be a graph with vertices $v_1, v_2, \dots, v_n$ and adjacency matrix A.

   (i) Define B to be the $n \times n$ matrix whose (i, j)-entry is given by

$$B_{i,j} = A_{i,1}A_{1,j} + A_{i,2}A_{2,j} + \cdots + A_{i,n}A_{n,j}.$$

   Show that $B_{i,j}$ is the number of walks of length 2 from v_i to v_j.

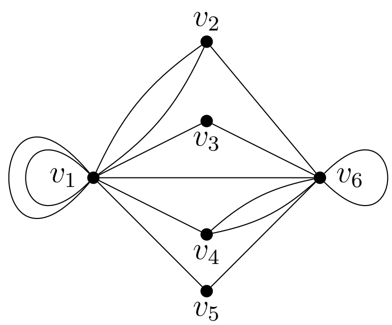
   Hint: Remember that $A_{i,k}$ is the number of edges from v_i to v_k. With this in mind, how can we interpret each product $A_{i,k}A_{k,j}$?

(ii) With B as above, define C to be the $n \times n$ matrix with (i,j)-entry

$$C_{i,j} = B_{i,1}A_{1,j} + B_{i,2}A_{2,j} + \cdots + B_{i,n}A_{n,j}.$$

Show that $C_{i,j}$ is the number of walks of length 3 from v_i to v_j.

(b) A graph G and its corresponding adjacency matrix A are given below. Use the results of part (a) to compute the number of walks of length 2 from v_1 to v_6.



$$A = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

5. Let G be a graph. If G is not connected, show that there is some way to order the vertices of G so that its adjacency matrix has the form

$$A = \left[\begin{array}{cccc|cccc} A_{1,1} & A_{1,2} & \cdots & A_{1,m} & 0 & 0 & \cdots & 0 \\ A_{2,1} & A_{2,2} & \cdots & A_{2,m} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,m} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & A_{m+1,m+1} & A_{m+1,m+2} & \cdots & A_{m+1,n} \\ 0 & 0 & \cdots & 0 & A_{m+2,m+1} & A_{m+2,m+2} & \cdots & A_{m+2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_{n,m+1} & A_{n,m+2} & \cdots & A_{n,n} \end{array}\right]_{n \times n}$$

for some integer m with $1 \leq m < n$. Explain why this is never possible when G is connected.

6. For each sequence d below, use the Havel-Hakimi algorithm to draw a simple graph with degree sequence d or show that such a graph does not exist.

(a) $d = (3, 3, 3, 2, 1)$

(b) $d = (4, 4, 4, 2, 2)$

(c) $d = (5, 4, 3, 2, 2, 2, 2, 2)$

7. Explain why the backtracking process following the Havel-Hakimi algorithm always produces a simple graph.