Intermediate Math Circles
Wednesday, February 8, 2017
Graph Theory I

Many of you are probably familiar with the term "graph". To you a graph may mean a line or curve defined by a function $y = f(x)$. It may also remind you of data graphs, such as bar graphs, line graphs, or pie charts. For the next few weeks we will be focussing on a different notion of graph that has very little in common with the types of graphs we already know and love.

1 What is a graph?

Informally, a **graph** is nothing more than a network of dots and lines. We call the dots **vertices** and call the lines **edges**. A graph may be represented by a picture or by listing all its vertices and edges. Here is an example:

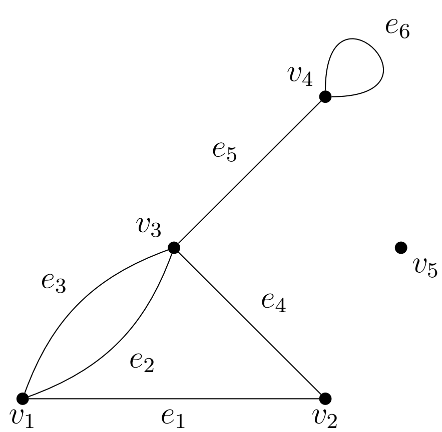

Figure 1: Graphical representation

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

$$E = \{e_1, e_2, e_3, e_4, e_5\}$$

$e_1 = \{v_1, v_2\}$
$e_2 = \{v_1, v_3\}$[1]
$e_3 = \{v_1, v_3\}$[2]
$e_4 = \{v_2, v_3\}$
$e_5 = \{v_3, v_4\}$
$e_6 = \{v_4, v_4\}$

Figure 2: Set representation

Notice that the graphs described in Figures 1 and 2 above are actually the same! One representation is given pictorially and the other is given by explicitly listing the vertices and telling us how to connect edges between them. Another thing worth mentioning is that the picture in Figure 1 could have been drawn in many different ways (for example, vertex v_5 could have been drawn on the left side of the graph, or we could have put v_1 at the top and v_6 at the bottom.) What's important is that the vertices and edges match the list from Figure 2. We will almost always take the Figure 1 approach for representing graphs. The picture route is great for humans while the list route is better for computers.

2 Where can graphs be used?

Now that we have seen our first graph, let's try to get a glimpse at why they are useful. Graphs are excellent tools for modelling networks and relationships. A social network is one such example: we could define a graph that represents all of Facebook by giving each profile its own vertex and connecting two profiles if the owners are mutual friends. We could model webpages on the internet by representing each page with a vertex and connecting two pages if there is a hyperlink from one to the other. Other applications of graph theory include matching problems (e.g., matching job applicants to available positions, or pairing people up as roommates) and colouring problems (e.g., what is the fewest number of colours need to colour a map of Canada?) Graphs can also be used to solve puzzles! Here is one of my favourites. Given the layout of islands and bridges in Figure 3, is it possible to start on one island, cross every bridge exactly once, and end at your original starting point? Another related problem is the following: Without lifting your pencil, can you draw a loop that runs through each door in Figure 4 exactly once? We will answer these questions in the coming weeks using graph theory.

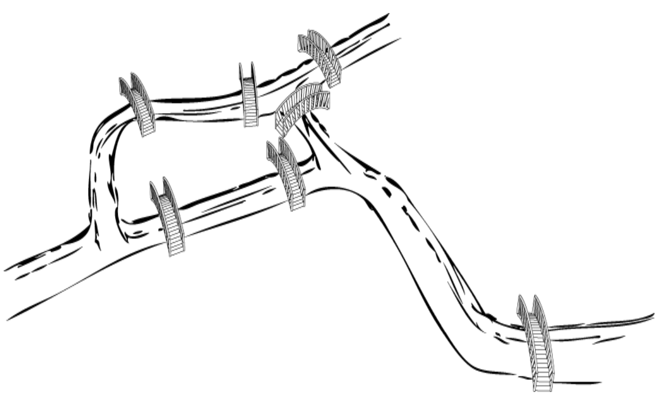

Figure 3: Bridge crossing problem

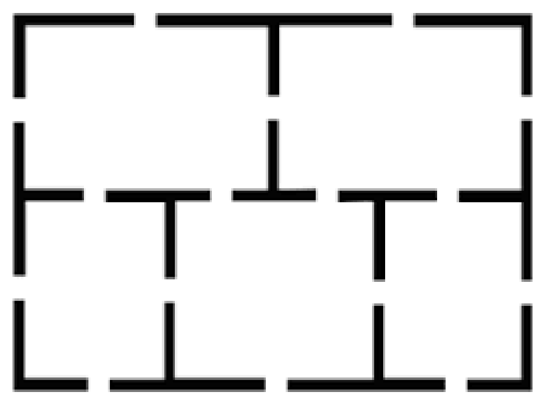

Figure 4: Door problem

3 Graph Theory Lingo

In this section we will introduce some terminology that is commonly used when discussing properties of graphs.

Definition. *Let G be a graph. A **walk** is an alternating list*

$$(v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_{k-1}, v_k)$$

*of vertices and edges that begins and ends with a vertex, and for each $i = 1, 2, \ldots, k$, we have e_i connects v_{i-1} to v_i. The **length** of a walk is the number of edges in the walk. A walk in which no vertex is repeated is called a **path**.*

*We say a graph is **connected** if any two distinct vertices in G can be connected by a path, and **disconnected** otherwise.*

Given a vertex v of G, we define the ***degree*** *of v to be the number of edges that enter or exit from v, and we denote this number by* $\deg(v)$. *Note that a* ***loop*** *(i.e., an edge from a vertex to itself) will contribute 2 to the degree of its vertex.*

For example, consider the following graph G:

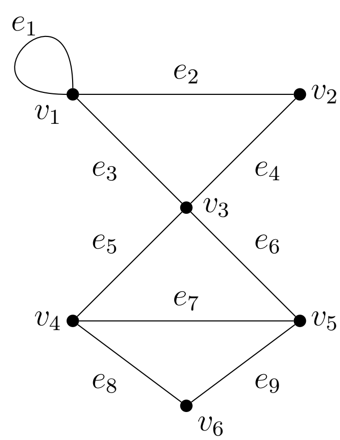

Figure 5: A connected graph

An example of walk in G of length 5 is

$$(v_2, e_4, v_3, e_6, v_5, e_7, v_4, e_5, v_3, e_3, v_1)$$

and an example of a path in G of length 3 is

$$(v_2, e_4, v_3, e_6, v_5, e_7, v_4).$$

We have that $\deg(v_1) = 4$ and $\deg(v_4) = 3$. What are the degrees of the remaining vertices? Note that this graph is connected as there is a path between any two distinct vertices. In Figure 1, no path exists between v_1 and v_5, and hence this is an example of a disconnected graph.

As you may have suspected there is a connection between the degrees of the vertices of a graph G and the total number of edges of G. It might be tempting to think that adding up the degrees will give the total number of edges in the graph, but this will lead to over counting. Indeed, by adding the degrees we will count each edge twice! Since each edge is incident to *two* vertices (or the same vertex twice in the case of a loop), the edge will be counted in the degree total for each of these vertices.

Proposition (Handshaking Theorem). *If G is a graph with vertices $v_1, v_2, \ldots, v_n$, then*

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = 2m$$

where m is the total number of edges in G.

This "handy" result gets its name from the following analogy. Suppose that there is a collection of guests at a party. Suppose that some guests shake hands with some other guests. If we asked everyone at the party how many guests they shook hands with and then added our findings, we would obtain exactly twice the total number of handshakes.

4 Families of Graphs

Here we will introduce some common families of graphs. Many of the graphs shown in this section are examples of **simple graphs** – they do not have loops and have at most one edge between any distinct pair of vertices. In Figure 1, edge e_6 is a loop and there are multiple edges e_2 and e_3 between vertices v_1 and v_3. For these reasons, the graph in Figure 1 is not simple. Such graphs are sometimes called **multigraphs**.

(1) The most basic example of a graph is a **null graph**. The null graph N_n on n vertices has no edges at all!

Figure 6: Null graph N_n on n vertices

(2) In contrast, the **complete graph** on n vertices, K_n, is a simple graph that contains an edge between every pair of distinct vertices. Here we have drawn K_n when $n = 3, n = 4$, and $n = 5$.

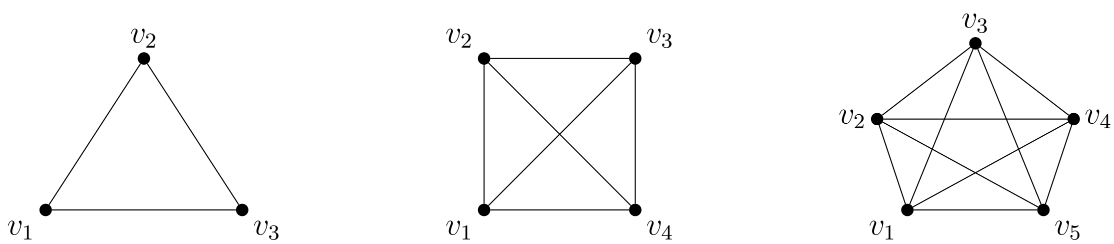

Figure 7: The complete graphs K_3, K_4, and K_5

(3) For $n \geq 3$, we define the **cycle graph** C_n to be the graph with vertices $v_1, v_2, \ldots, v_n$ and an edge between v_i and v_{i+1} for every i, as well as an edge from v_n to v_1. Below you will find C_n when $n = 3, n = 4$, and $n = 5$.

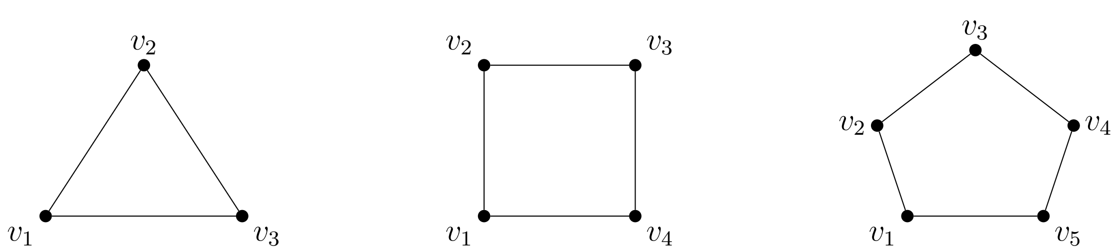

Figure 8: The cycle graphs C_3, C_4, and C_5

(4) For each $n \geq 4$, we can consider the **wheel graph** W_n on n vertices. It looks just like the cycle graph C_{n-1}, but with every vertex joined by an edge to an additional vertex in the centre. Here is W_n when $n = 4, n = 5$, and $n = 6$.

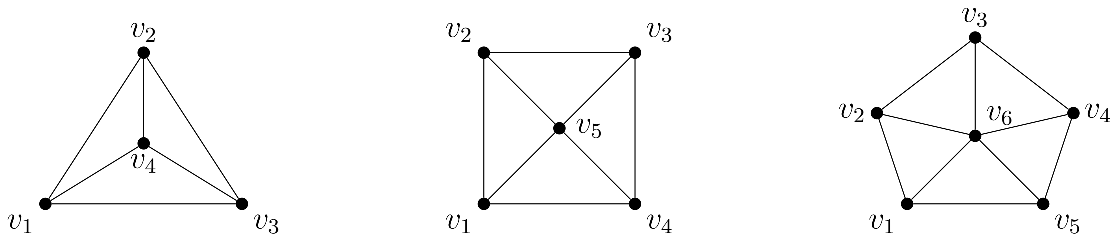

Figure 9: The wheel graphs W_4, W_5, and W_6

(5) For each integer $n \geq 1$, the **star graph** S_n consists of $n-1$ vertices each connected by a single edge to a centre vertex. We can think of this as simply the wheel graph W_n with the outer edges removed. Here is S_n when $n = 4, n = 5$, and $n = 6$.

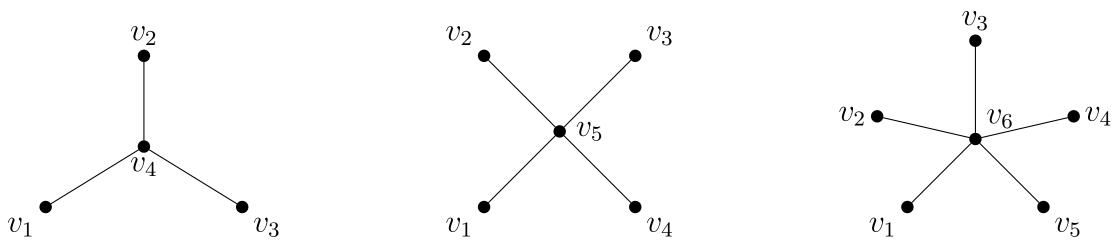

Figure 10: The star graphs S_4, S_5, and S_6

(6) A **bipartite**[1] **graph** is a graph G whose vertices can be grouped into two collections $v_1, v_2, \ldots, v_n$ and $w_1, w_2, \ldots, w_m$ so that every edge of G connects a vertex in the first collection to a vertex in the second collection. This is best understood through some examples.

Each of the graphs in Figure 11 is bipartite. Indeed, edges take v's to w's and w's to v's. If we were to add another edge from v_2 to v_3 in the second graph below, the new graph would fail to be bipartite.

A **complete bipartite graph** is a simple bipartite graph G such that every vertex in the first group is connected to every vertex in the second group. A complete bipartite graph with n vertices in one group and m vertices in the other is often denoted $K_{n,m}$. As an exercise, which of the graphs that we have seen so far are bipartite? Which of these are complete bipartite?

[1]Bipartite means *consisting of two parts.*

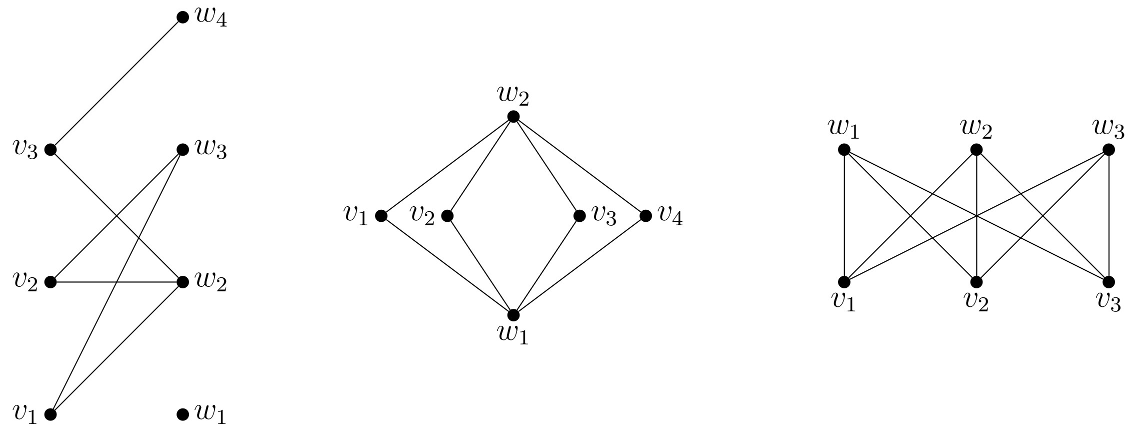

Figure 11: Three bipartite graphs

(7) Our final example is of a family of graphs in which every vertex has the same degree. For a fixed integer $k \geq 0$, a graph G is called **k-regular** if each vertex of G has degree k. We have seen examples of these already! The null graph N_n on n vertices is a 0-regular graph, while the cycle graph C_n on n vertices is 2-regular. The complete graph K_n is $(n-1)$-regular.