Intermediate Math Circles
Wednesday, February 8, 2017
Problem Set 1

1. (a) Draw a graph whose vertices and edges are given by

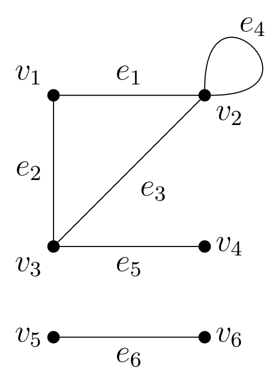
$$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$$
$$E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

$$e_1 = \{v_1, v_2\}$$
$$e_2 = \{v_1, v_3\}$$
$$e_3 = \{v_2, v_3\}$$
$$e_4 = \{v_2, v_2\}$$
$$e_5 = \{v_3, v_4\}$$
$$e_6 = \{v_5, v_6\}$$

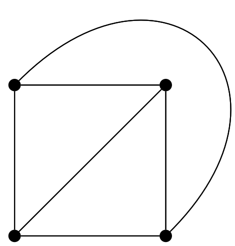
(b) Draw a simple graph G with 4 vertices and 6 edges. Find a way to draw this graph so that no two edges cross.

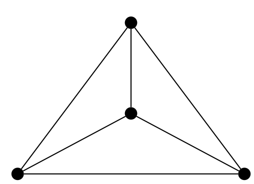
(c) Draw a simple connected graph with 6 vertices and 7 edges such that the removal of one of the edges disconnects the graph.

Solution: (a) Place vertices $v_1, v_2, \ldots, v_6$ in any orientation and then connect edges according to the list above. Your graph may look something like this:

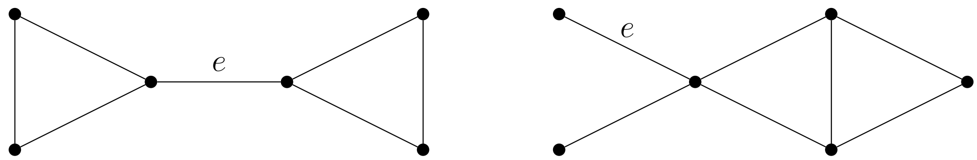


(b) Here are two possible examples. There are probably many more interesting ways these can be drawn. The example on the right has the nice feature that all edges are straight lines.

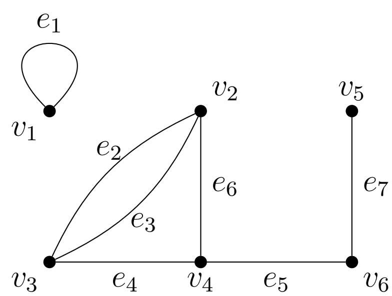




(c) The bow tie and fish are two possible examples. Notice that the removal of the edge e in either graph will lead to a disconnection.



2. In the following graph G, find an example of a path of length 4, and an example of a walk of length 3 that is *not* a path. List the degree for each vertex in G. Is G connected?



Solution: An example of a path in G of length 4 is

$$(v_5, e_7, v_6, e_5, v_4, e_4, v_3, e_3, v_2),$$

and an example of a walk of length 3 that is not a path is

$$(v_2, e_2, v_3, e_3, v_2, e_6, v_4).$$

The degree sequence for G is

$$\deg(v_5) = 1, \deg(v_1) = \deg(v_6) = 2, \deg(v_2) = \deg(v_3) = \deg(v_4) = 3.$$

It is easy to see that G is not connected, as there is no path from v_1 to v_2 (or from v_1 to *any* of the other vertices of G, for that matter).

3. Determine the number of edges in each of the following graphs:

(a) N_n

(b) C_n with $n \geq 3$

(c) W_n with $n \geq 4$

(d) S_n

(e) K_n

(f) $K_{m,n}$ with $m, n \geq 1$

(g) A k-regular graph on n vertices.

Solution: (a) The null graph N_n has no edges at all!

(b) In the cycle graph C_n, every vertex is connected to the one immediately before it and immediately after it, so $\deg(v_i) = 2$ for each $i = 1, 2, \ldots, n$. By the handshaking theorem, we have that

$$2m = \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) = \underbrace{2 + 2 + \cdots + 2}_{n \text{ times}} = 2n$$

where m is the number of edges in C_n. Dividing by 2, we see that $m = n$, so C_n has n edges.

(c) The wheel graph W_n looks a lot like the cycle graph, but with every vertex connected to one additional vertex in the middle. That said, W_n has $n - 1$ edges (by applying part (b) to the $n - 1$ outer vertices) plus an additional edge for each outer vertex. This means W_n has

$$(n - 1) + (n - 1) = 2n - 2$$

edges.

(d) One could use the handshaking theorem to argue as in part (b). Alternatively, we mentioned in the notes that the star graph S_n looks like the wheel graph W_n with the outer edges removed. Thus, there are $n - 1$ edges to be removed, as they form the cycle graph C_{n-1} on the outer $n - 1$ vertices. Since W_n has $2n - 2$ edges (by (c)), it must be the case that S_n has

$$(2n - 2) - (n - 1) = n - 1$$

edges.

(e) Every vertex in K_n is connected to every other vertex. This means that each of the n vertices has $n - 1$ edges. By the handshaking theorem,

$$\begin{aligned} 2m &= \deg(v_1) + \deg(v_2) + \cdots + \deg(v_n) \\ &= \underbrace{(n - 1) + (n - 1) + \cdots + (n - 1)}_{n \text{ times}} = n(n - 1) \end{aligned}$$

where m is the number of edges in K_n. Hence, $m = \frac{n(n-1)}{2}$ edges.

(f) In $K_{n,m}$, each of the n vertices in the first vertex set is connected to every vertex in the second vertex set, so each has degree m. Likewise, each of the m vertices in the second set has degree n. We again use the handshaking theorem to see that

$$\begin{aligned} 2p &= (\text{sum of degrees in first vertex set}) + (\text{sum of degrees in second vertex set}) \\ &= nm + mn \\ &= 2mn \end{aligned}$$

where p is the number of edges in $K_{n,m}$. Thus, $K_{n,m}$ has $p = mn$ edges.

(g) If G is k-regular, then by definition $deg(v) = k$ for all vertices v. Thus, the sum of the degrees of all n vertices in G is nk. The handshaking theorem tells us this is exactly twice the number of edges, and hence the number of edges in G is $nk/2$.

Remark. *An efficient approach to the above problem would be to start with parts (f) and (g). Once these are solved, we can tackle (a),(b),(d), and (e) by noting that*

- N_n *is a* 0*-regular graph,*
- C_n *is a* 2*-regular graph,*
- $S_n = K_{1,n}$ *is a complete bipartite graph, and*
- K_n *is an* $(n-1)$*-regular graph.*

4. For each graph N_n, C_n, W_n, and S_n, determine the values of n for which the graph is bipartite.

Solution: The null graph N_n is bipartite for any n. If there are no edges, we can divide the vertices into two groups in any way we wish. Note that one or both of these groups will be empty if $n \leq 1$.

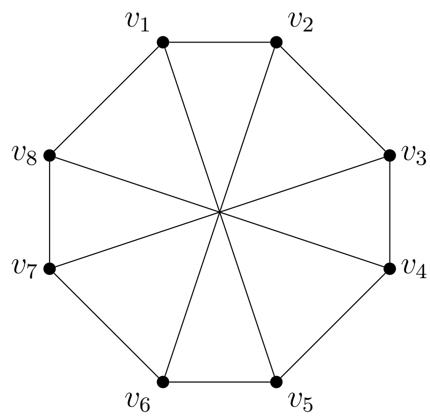
The cycle graph is bipartite precisely when n is even. To see this, label the vertices $v_1, v_2, \ldots, v_n$ so that v_j is connected to v_{j+1} for all j, and v_n connects back to v_1. If n is even, we may place the vertices with even index into the first group, and place the vertices with odd index into the second group to arrive at a bipartition. This method won't be so fruitful when n is odd. Draw a few examples to see why.

The wheel graph W_n will never be bipartite. Indeed, if it were, then we could find a bipartition and we may assume that the central vertex belongs to the first group. Since it connects to every other vertex, each of the other vertices has to belong to the second group. Uh oh... some of the non-central vertices are joined to each other. This will ruin our shot at a bipartite graph.

Finally, we examine the star graph S_n. This graph is always bipartite. To see this, simply let the centre vertex be in a group by itself, and let all other vertices make up the second group. Since the only edges of S_n are from the centre vertices to the non-centre ones, we have made a valid bipartition.

5. Given a positive integer k, explain how one would construct a 3-regular graph on $2k$ vertices. Draw such a graph when $k = 4$.

Solution: Label the vertices $v_1, v_2, \ldots, v_{2k}$. Start by considering the cycle graph C_{2k} on these vertices. Here every vertex has degree 2. To bump each degree up to three, simply match each vertex to the one opposite it. That is, insert edges between v_1 and v_{k+1}, v_2 and v_{k+2}, v_3 and v_{k+3}, etc. Finish by inserting an edge between v_k and v_{2k} to arrive at a 3-regular graph. Below is an example when $k = 4$:



6. Why must every graph have an even number of vertices of odd degree?

Solution: Suppose that the number of vertices of odd degree is itself odd. If these vertices are labelled $v_1, v_2, \ldots, v_k$, and the remaining (even degree) vertices are labelled $v_{k+1}, v_{k+2}, \ldots, v_n$, then the handshaking theorem tells us that

$$2m = (\deg(v_1) + \deg(v_2) + \cdots + \deg(v_k)) + (\deg(v_{k+1}) + \deg(v_{k+2}) + \cdots + \deg(v_n))$$

This means that

$$\deg(v_1) + \deg(v_2) + \cdots + \deg(v_k) = 2m - (\deg(v_{k+1}) + \deg(v_{k+2}) + \cdots + \deg(v_n)).$$

Do you see the problem? The sum on the left is odd, because it is a sum of an odd number of odd integers. However, the sum on the right is even, as it is a sum/difference of even numbers. But the left and right sides must be equal, so this is a contradiction. It must therefore be the case that the number of vertices of odd degree is even.

7. Construct a simple graph on $n \geq 2$ vertices such that no two vertices have the same degree or argue that such a graph cannot exist. What if the graph is not simple?

Solution: Such a graph does not exists. To see this, let G be a simple graph on n vertices where $n \geq 2$. To reach a contradiction, suppose that no two vertices have the same degree. What are the possible degrees for the vertices of G? Since the graph is simple, any given vertex must have one of the following degrees: $0, 1, 2, \ldots, n-1$.

Notice that since there are n possibilities listed above and G has n vertices, the only way for every vertex to have distinct degree is if there is exactly one vertex for each of the above choices. That is, one vertex of G must have degree 0, one must have degree 1, one must have degree 2, etc. In particular, there is a vertex of G with degree $n-1$, and hence this vertex must be connected to every other vertex. But we said that some vertex of G has degree 0, meaning it is connected to no vertices at all! This is a contradiction and therefore there must be at least two vertices of the same degree.

8. Let G be a simple graph on $n \geq 2$ vertices. Suppose that for every pair of distinct vertices u, v in G, we have

$$\deg(u) + \deg(v) \geq n - 1.$$

Show that G must be connected.

Solution: If $n = 2$ then G is simply two vertices connected by an edge, which is obviously a connected graph. On this note we may assume that $n \geq 3$ and let u and v be two distinct vertices in G. If they are connected to each other, that's great! Otherwise, the fact that $\deg(u)+\deg(v) = n-1$ means that there are $n-1$ vertices that are connected to u or to v, and hence one of the remaining $n-2$ vertices of G, say w, must be connected to both u and v. This proves that there is a path from u to w and from w to v, and therefore u and v are connected by a path. Since these vertices were arbitrary, any two vertices of G are connected by a path and we deduce that G is connected.