# Intermediate Math Circles
# Wednesday, March 1, 2017
# Problem Set 4

1. Find the tens digit of $2^{2017}$.

   **Solution:**

   To find the tens digit of $2^{2017}$ we make a table of the last two digits of $2^i$. (For the single digit powers we will consider the power to start with the digit 0.)

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
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| $2^i$ | 02 | 04 | 08 | 16 | 32 | 64 | 28 | 56 | 12 | 24 | 48 | 96 | 92 | 84 | 68 | 36 | 72 | 44 | 88 | 76 | 52 | 04 |

   We notice that the last two digits of $2^{22}$ are the same as the last two digits of $2^2$. Therefore, from this point onward the last two digits will cycle through a sequence which has 20 terms. (Notice that the last two digits of $2^1$ are not part of this repeating sequence.)

   Since $2017 = 1 + 100(20) + 16$, then we need to pick out the sixteenth term in the repeated sequence. This term is 72. (It is the last two digits of $2^{17}$.) Therefore, the tens digit of $2^{2017}$ is 7.

2. The sequence of numbers $a_1, a_2, a_3, \ldots$ satisfies $\dfrac{a_m}{a_n} = \dfrac{m}{n}$ for every pair of positive integers $m$ and $n$. If $a_3 = 5$, evaluate $3a_{20}$.

   **Solution:** Since $\dfrac{a_m}{a_n} = \dfrac{m}{n}$, then $\dfrac{a_3}{a_{20}} = \dfrac{3}{20}$. But we also know that $a_3 = 5$. So, $\dfrac{5}{a_{20}} = \dfrac{3}{20}$. Therefore $3a_{20} = 5(20) = 100$.

3. Find a geometric sequence and an arithmetic sequence that have the same first three terms.

   **Solution:** Consider a geometric sequence that begins $a, ar, ar^2$, where $r$ is the common ratio. Consider an arithmetic sequence that begins $a, a+d, a+2d$ where $d$ is the common difference. Since the first three terms of the two sequences match, then $a = a$, $ar = a + d$ and $ar^2 = a + 2d$.

   If $a = 0$, then the geometric sequence is a sequence where every term is 0. In this situation, the common ratio could be any value. We also notice that this we can think of this sequence as an arithmetic sequence where the common difference is 0. Therefore, the sequence where every term is 0 is both a geometric and an arithmetic sequence and we've solved the problem.

   Is this sequence the only solution? Let's assume that $a \neq 0$ and return to our equations.

   We multiply both sides of the second equation by 2 and then subtract the third equation to obtain $2ar - ar^2 = a$. Since $a \neq 0$ we can divide both sides by $a$ to obtain $2r - r^2 = 1$,. Rearranging we obtain the equation $r^2 - 2r + 1 = 0$. This factors as $(r-1)^2 = 0$, and therefore the only solution is $r = 1$. If $r = 1$, then our geometric sequence is a constant sequence where every term is $a$. We notice that a constant sequence can be thought of as an arithmetic sequence where the common difference, $d$, is 0. And if we use the fact that $r = 1$, then our second equation does in fact give us that $d = 0$.

   Therefore, if a geometric sequence and an arithmetic sequence have the same first three terms, then they are both in fact a constant sequence, where every term in the sequence is $a$, the value of the first term.

4. Consider the sequence $t_1 = 1$, $t_2 = -1$ and $t_n = \left(\frac{n-3}{n-1}\right) t_{n-2}$ where $n \geq 3$. What is the value of $t_{2016}$?

**Solution:**

Using the formula for $t_n$, we calculate the next few terms in the sequence. Thus, $t_3 = 0$, $t_4 = -\frac{1}{3}$, $t_5 = 0$, $t_6 = -\frac{1}{5}$, $t_7 = 0$, $t_8 = -\frac{1}{7}$. Using pattern recognition we can see that the even terms are 0 and the odd terms are $-\frac{1}{n-1}$. Therefore $t_{2016} = -\frac{1}{2015}$.

**Another Solution:**

Using the definition of $t_n$ we obtain

$$\begin{aligned} t_{2016} &= \left(\frac{2013}{2015}\right) t_{2014} \\ &= \frac{2013}{2015}\left(\frac{2011}{2013}\right) t_{2012} \\ &\vdots \\ &= \frac{2013}{2015}\left(\frac{2011}{2013}\right) \cdots \left(\frac{3}{5}\right)\left(\frac{1}{3}\right) t_2 \\ &= \left(\frac{1}{2015}\right) t_2 \end{aligned}$$

Since $t_2 = -1$, then $t_{2016} = -\frac{1}{2015}$.

5. In a sequence, every term after the second term is twice the sum of the two preceding terms. The seventh term of the sequence is 8, and the ninth term is 24. What is the eleventh term of the sequence?

**Solution:**

This sequence is a recursive sequence with the following definition: $t_n = 2(t_{n-1} + t_{n-2})$ for $n \geq 3$.

Therefore,

$$\begin{aligned} t_9 &= 2(t_8 + t_7) \\ 24 &= 2t_8 + 2(8) && \text{(Since } t_9 = 24 \text{ and } t_7 = 8\text{)} \\ 8 &= 2t_8 \\ t_8 &= 4 \end{aligned}$$

Using the recursive definition of the sequence we calculate $t_{10} = 2(24 + 4) = 56$ and $t_{11} = 2(56 + 24) = 160$.

6. Consider the sequence defined by $t_n = n^2$. Find a recursive definition for this sequence. (There is more than one answer.)

**Solution:**

The sequence starts as follows $1, 4, 9, 16, 25, 36, \ldots$.

The difference between consecutive terms forms the sequence $3, 5, 7, 9, 11, 13, \ldots$. This sequence is the odd integers starting at 3.

Therefore one possible recursive definition for the given sequence is $t_1 = 1$, $t_n = t_{n-1} + 2n - 1$ for $n \geq 2$.