Intermediate Math Circles
Wednesday, March 22, 2017
Analytic Geometry I

Addition Magician

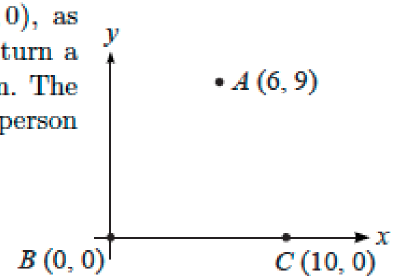
Two players begin a game with a piece of paper and a pencil. Players alternate turns, each player adding 1, 2, 3, or 4 to the total on their turn and whoever goes first starts with a total of 0. The winner is the player who brings to total to 33. Is there a winning strategy? Is the player going first or the player going second guaranteed to win with this strategy?

Double Trouble

Two players begin a game with two separate bags of soccer balls, one bag with 13 balls and the other with 17. Players alternate turns and on each turn a players removes 1, 2, 3, or 4 balls from one of the two bags. A player can only remove balls from one bag. The winner is the person who after their turn both bags are empty. Is there a winning strategy? Is the player going first or the players going second guaranteed to win with this strategy?

2014 Fryer - Question 3

Triangle ABC begins with vertices $A(6,9), B(0,0), C(10,0)$, as shown. Two players play a game using $\triangle ABC$. On each turn a player can move vertex A one unit, either to the left or down. The x- and y-coordinates of A cannot be made negative. The person who makes the area of $\triangle ABC$ equal to 25 wins the game.



(a) What is the area of $\triangle ABC$ before the first move in the game is made?

(b) Dexter and Ella play the game. After several moves have been made, vertex A is at $(2,7)$. It is now Dexter's turn to move. Explain how Ella can always win the game from this point.

(c) Faisal and Geoff play the game, with Faisal always going first. There is a *winning strategy* for one of these players; that is, by following the rules in a certain way, he can win the game every time no matter how the other player plays.

(i) Which one of the two players has a winning strategy?

(ii) Describe a winning strategy for this player.

(iii) Justify why this winning strategy described in (ii) always results in a win.

What's So Special About a Math Game?

Example of **Game Theory**, which is widely used in economics, politics, evolution, psychology

Business, Accounting, and Finance

- The world of business and finance is a competitive place
- Mathematics and computer science are what make businesses competitive and accountable
- One of the ways you can set yourself apart is by developing that "mathematical tool belt"
- Not just the technical skills, but also your problem solving skills

What is Analytic Geometry?

- Also called "Coordinate Geometry" or "Cartesian Geometry"
- There are two branches
  1. **Plane Analytic Geometry** which deals with points, lines, and curves restricted to a plane (two dimensional space)
  2. **Solid Analytic Geomerty** which deals with points, lines, planes, curves, and surfaces in a three dimensional space.
- Developed largely by René Descartes in the 1600s and brought about lots of advantages to solving geometric problems
- Deals with solving geometric problems by algebraic methods

Why Should You Care?

Coordinates are everywhere!

- Navigation
- Design
- Construction
- Physics
- Chemistry
- Think about how often you see graphs

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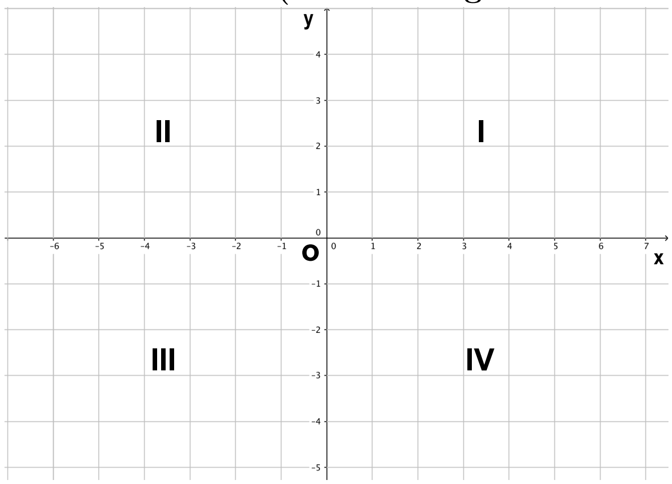
Review of Line Segments in R^2

Cartesian Coordinates (aka Rectangular Coordinates)



Review of Lines in R^2

Points & Line Segments

- Any point P(x,y)
- Line Segment AB

- Distance Formula
  Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the length of line segment AB

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- Midpoint Formula
  Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the length of line segment AB

$$\mathrm{M}\left(\frac{(x_1 + x_2)}{2}, \frac{(y_1 + y_2)}{2}\right)$$

Distance Formula Notation Given points $A(x_1, y_1)$ and $B(x_2, y_2)$, then the length of line segment AB

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

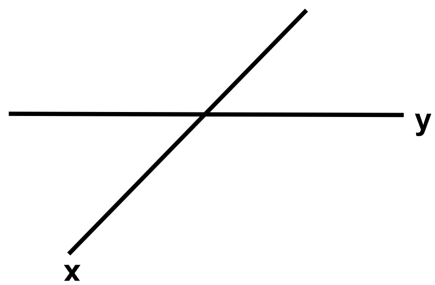
Notation:

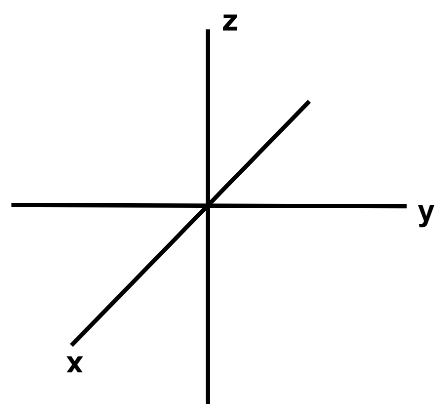
- AB
- $|AB|$
- d_{AB}
- $d(A, B)$
- $|\overline{AB}|$

So Long Flatland... Hello R^3

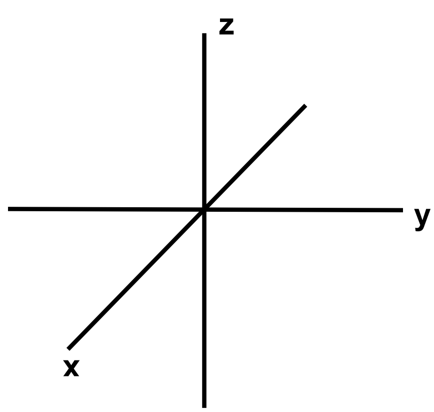
In R^2 we have an x- and y-axes.

In R^3 we have an x-, y-, and z-axes.





R^3 You Beautiful Axes You



Any point in R^3 has an x, y, and z component.

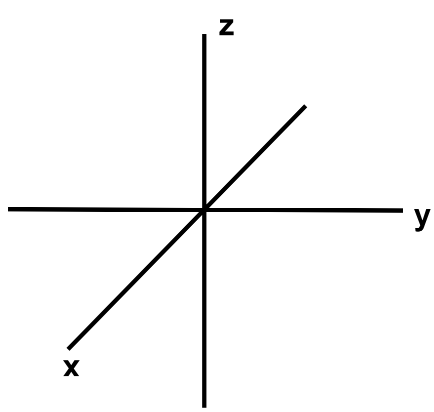
I.e. $(x, y, z) \in R^3$.

Where is $(0, 0, 0)$?

Where is $(3, 0, 0)$?

Where is $(0, -2, 0)$?

Where is $(0, 0, 1)$?



Where is $(1, 4, 0)$?

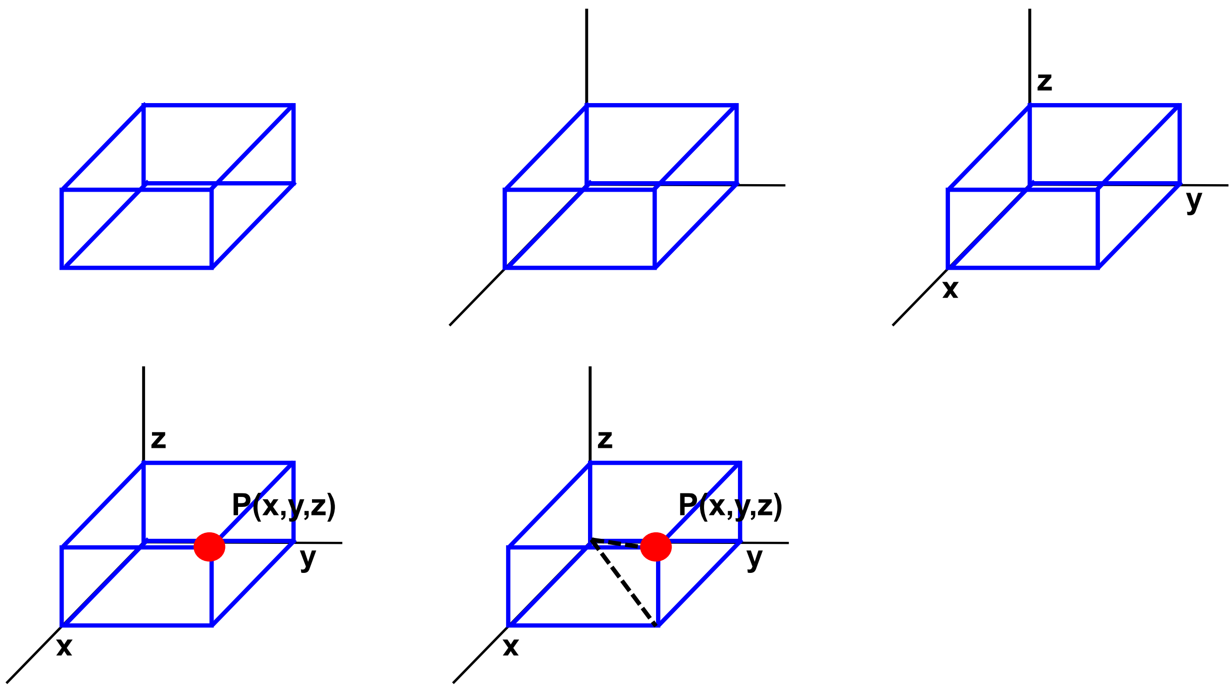
Where is $(3, 0, -2)$?

Where is $(0, 1, 1)$?

Where is $(2, -1, 1)$?

Going the Distance

In R^2 we know $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, but what about measuring distances in R^3?



In R^3 we just found out the distance from $(0, 0, 0)$ to any point (x, y, z) is

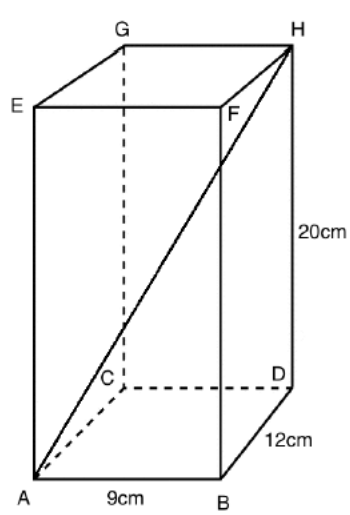
$$d = \sqrt{x^2 + y^2 + z^2}$$

What about for any two given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$?

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example [Turn the Crank]

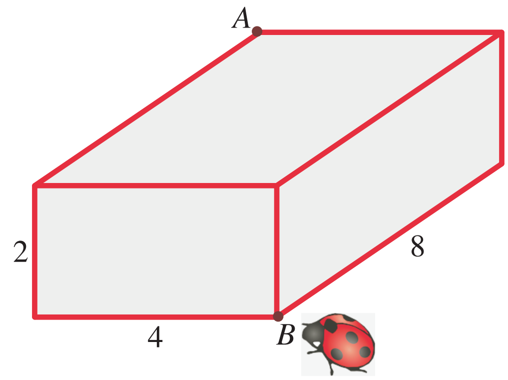
Given the rectangular prism below, find the length of line segment AH.



Example [The Crank is Broken!]

A ladybug wishes to travel from B to A on the surface of a wooden block with dimensions 2 by 4 by 8 as shown in the diagram.

Determine the shortest distance for the ladybug to walk.



Example [Finding the Middle Ground]:

Find the coordinates of the midpoint of points C(-4,7) and D(1,-8).

Example [Internal Division of a Line Segment]:

Find the coordinates of the point N dividing the distance between C(-4,7) to D(1,-8) internally in the ratio 2 : 3.

Example [Internal Division of a Line Segment in General]:

Find the coordinates of the point N dividing the distance between C(-4,7) to D(1,-8) internally in the ratio $a : b$.

Challenge [Develop a Formula]:

Come up with a formula for dividing the distance between (x_1, y_1) to (x_2, y_2) internally into the ratio $a : b$.

Review of Lines in R^2

- Lines contain infinitely many line segments

- Slope
  - Slope measures steepness and direction of a line (upward or downward)
  - Given $A(x_1, y_1)$ and $B(x_2, y_2)$ where $x_1 \neq x_2$

    $$\text{slope} = m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$
  - Slope of a horizontal line is 0
  - Slope of a vertical line is undefined
  - When lines are parallel $m_1 = m_2$
  - When lines are perpendicular $m_2 = -\frac{1}{m_1}$

Example [Diagonals of a Rectangle]:

a.) Show that the diagonals of a rectangle bisect each other

b.) Determine under what conditions will the diagonals of a rectangle be perpendicular bisectors of each other

Review of Lines in R^2

Equations of Lines

- Horizontal Line: $y = k$ where $k \in R$
- Vertical Line: $x = h$ where $h \in R$
- Slope-intercept Equation: $y = mx + b$
- General Equation: $Ax + By + C = 0$
- Points in intersections

  Case 1: no points of intersection

  Case 2: one point of intersection

  Case 3: infinitely many points of intersection
  (i.e. they are collinear)

Example [Distance from a Point to a Line]:

Given the line $\mathscr{L}$: $2x + y - 10 = 0$ and $P(-2, 9)$

a.) Find the point on $\mathscr{L}$, call it Q, closest to P

b.) State the equation of the line through PQ in slope-intercept form