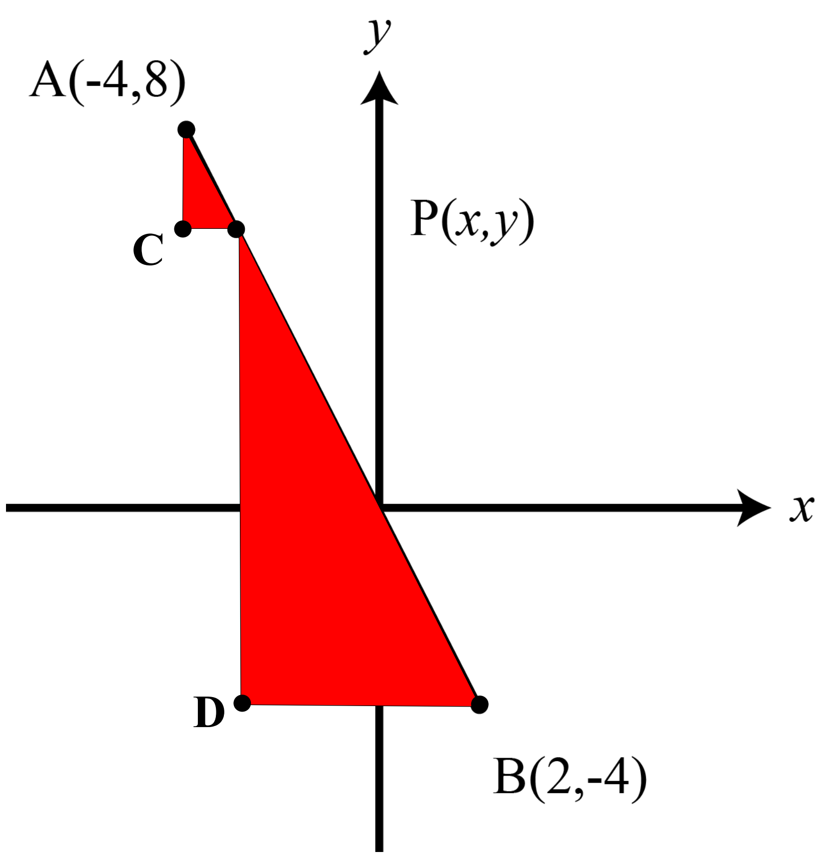
Intermediate Math Circles
Wednesday, March 22, 2017
Problem Set 6

1. Find the coordinates of the point that divides the distance

   (a) from $(7, 1)$ to $(3, 5)$ internally in the ratio 1:1 (i.e. find the midpoint)
   $(\frac{7+3}{2}, \frac{1+5}{2}) = (5, 3)$

   (b) from $(-4, 8)$ to $(2, -4)$ internally in the ratio 1:5
   Label $A(-4, 8)$ and $B(2, -4)$. We want some point $P(x, y)$ such that $\dfrac{d_{AP}}{d_{PB}} = \dfrac{1}{5}$.



Since the AP and PB could be the hypotenuses of similar right triangles according to the ratio 1:5 we can apply these ratios to the horizontal (x-value) and vertical (y-value) components of these triangles.

x-component

$$\begin{aligned} \frac{CP}{DB} &= \frac{1}{5} \\ \frac{x-(-4)}{2-x} &= \frac{1}{5} \\ \frac{x+4}{2-x} &= \frac{1}{5} \\ 5x+20 &= 2-x \\ 6x &= -18 \\ x &= -3 \end{aligned}$$

y-component

$$\begin{aligned} \frac{AC}{PD} &= \frac{1}{5} \\ \frac{y-8}{-4-y} &= \frac{1}{5} \\ 5y-40 &= -4-y \\ 6y &= 36 \\ y &= 6 \end{aligned}$$

Therefore, P(-3, 6) is the desired internal point.

(c) from $(-4, 8)$ to $(2, -4)$ internally in the ratio 2:1

Again, we want some point $P(x, y)$ such that $\frac{d_{AP}}{d_{PB}} = \frac{2}{1}$.

x-component

$$\begin{aligned} \frac{x+4}{2-x} &= 2 \\ x+4 &= 4-2x \\ 3x &= 0 \\ x &= 0 \end{aligned}$$

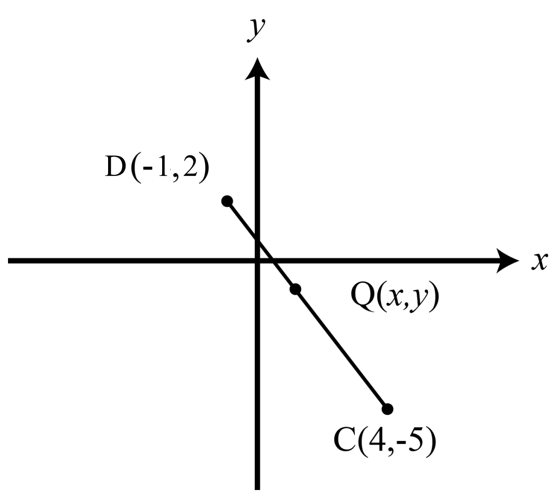
y-component

$$\begin{aligned} \frac{y-8}{-4-y} &= 2 \\ y-8 &= -8-2y \\ y &= 0 \end{aligned}$$

Therefore, P(0,0) is the desired internal point.

(d) from $(4, -5)$ to $(-1, 2)$ internally in the ratio 3:2

Label $C(4, -5)$ and $D(-1, 2)$. We want some point $Q(x, y)$ such that $\frac{d_{CQ}}{d_{QD}} = \frac{3}{2}$



x-component

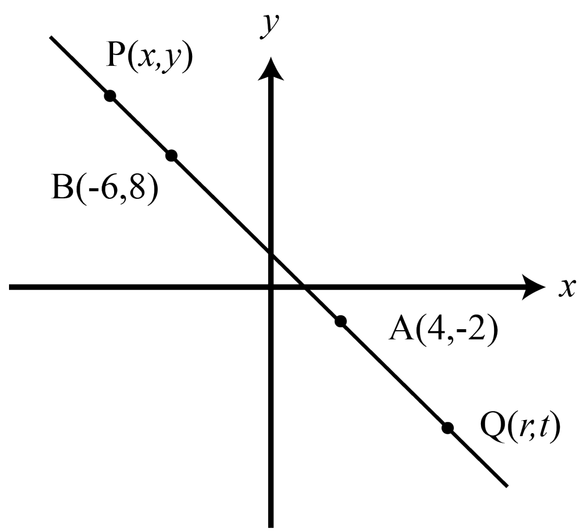
$$\begin{aligned}\frac{x-4}{-1-x} &= \frac{3}{2}\\ 2x-8 &= -3-x\\ 5x &= 5\\ x &= 1\end{aligned}$$

y-component

$$\begin{aligned}\frac{y-(-5)}{2-y} &= \frac{3}{2}\\ \frac{y+5}{2-y} &= \frac{3}{2}\\ 2y+10 &= 6-3y\\ 5y &= -4\\ y &= -\frac{4}{5}\end{aligned}$$

Therefore, $\mathrm{Q}(1, -\frac{4}{5})$ is the desired internal point.

2. (a) Find the coordinates of point P which divides the length from $A(4, -2)$ to $B(-6, 8)$ externally in the ratio of 3 : 1. By externally, I mean find point P on the same line as AB, but beyond the line segment AB.



We can some point $P(x, y)$ such that $\dfrac{d_{PA}}{d_{PB}} = \dfrac{3}{1}$. The ratio tells us that P is closer to B on $\mathscr{L}_{AB}$.

x-component

$$\begin{aligned}\frac{4-x}{-6-x} &= 3\\ 4-x &= -18-3x\\ 2x &= -22\\ x &= -11\end{aligned}$$

y-component

$$\begin{aligned}\frac{-2-y}{8-y} &= 3\\ -2-y &= 24-3y\\ 2y &= 26\\ y &= 13\end{aligned}$$

Therefore, the desired external point is $(-11, 13)$.

(b) For the same points A & B above, find the coordinates of point Q which divides the length of AB externally in the ratio of 1 : 3

Let $Q(r,t)$ be some point such that $\frac{d_{QA}}{d_{QB}} = \frac{1}{3}$.

x-component

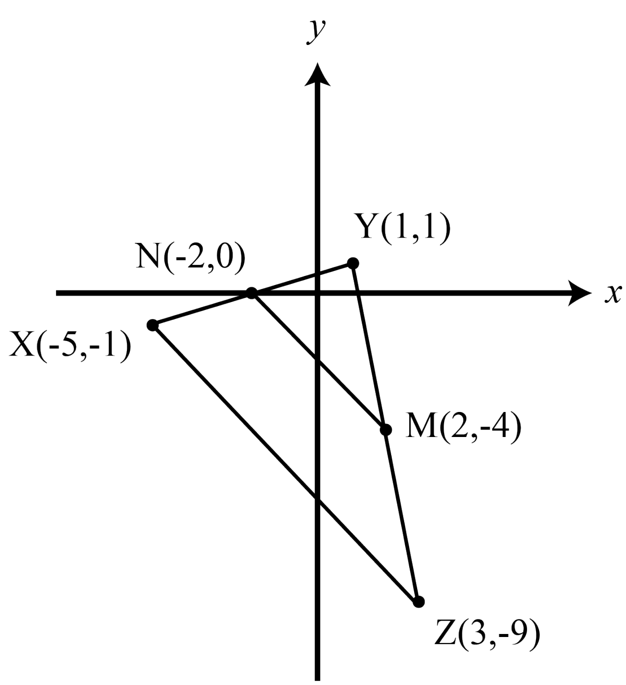
$$\begin{aligned}\frac{4-r}{-6-r} &= \frac{1}{3}\\ 12-3r &= -6-2\\ 18 &= 2r\\ r &= 9\end{aligned}$$

y-component

$$\begin{aligned}\frac{-2-t}{8-t} &= \frac{1}{3}\\ -6-3t &= 8-t\\ -14 &= 2t\\ t &= -7\end{aligned}$$

Therefore, the desired external point is $(9, -7)$.

3. Show that the line joining the midpoints of the two sides of triangle with vertex $(1, 1)$ is parallel and equal to half the base $(-5, -1)$, $(3, -9)$.



Find the midpoints of line segments XY and YZ.

$$\begin{aligned}N &= M_{XY}\left(\frac{-5+2}{2}, \frac{1-1}{2}\right)\\ &= (-2, 0)\\ M &= M_{YZ}\left(\frac{1+3}{2}, \frac{1-9}{2}\right)\\ &= (2, -4)\end{aligned}$$

Find the slopes of line segments XY and YZ.

$$\begin{aligned} m_{NM} &= \frac{-4-0}{2-(-2)} \\ &= -1 \\ m_{XY} &= \frac{3-(-5)}{-9-(-1)} \\ &= -1 \end{aligned}$$

Since $m_{NM} = m_{XY}$, therefore $NM||XY$.

Find the length of line segments XY and NM.

$$\begin{aligned} d_{XY} &= \sqrt{[3-(-5)]^2 + (-9-(-1))} \\ &= \sqrt{8^2+8^2} \\ &= 8\sqrt{2} \\ d_{NM} &= \sqrt{[2-(-2)]^2+(-4-0)^2} \\ &= \sqrt{4^2+4^2} \\ &= 4\sqrt{2} = \frac{1}{2}d_{XY} \end{aligned}$$

Therefore $d_{XY} = \frac{1}{2}d_{XY}$.

4. Given the points $A(-3,0)$, $B(-2,-5)$, $C(2,1)$, find a point P that $PA||BC$ and $PC \perp BC$.

Let $P(x,y)$ be our desired point.

We know the following facts about slopes of parallel and perpendicular lines. These facts can be used to find two equations in terms of x and y.

$\underline{PA||BC}$

$$\begin{aligned} m_{PA} &= m_{BC} \\ \frac{0-y}{-3-x} &= \frac{1-(-5)}{2-(-2)} \\ \frac{-y}{-3-x} &= \frac{6}{4} = \frac{3}{2} \\ 2y &= 9+3x \\ 0 &= 3x-2y+9 \quad (2) \end{aligned}$$

$\underline{PC \perp BC}$

$$\begin{aligned} m_{PC} &= -\frac{1}{m_{BC}} \\ \frac{1-y}{2-x} &= -\frac{1}{\frac{3}{2}} \\ \frac{1-y}{2-x} &= -\frac{2}{3} \\ 3(1-y) &= -2(2-x) \\ 3-3y &= -4+2x \\ 0 &= 2x+3y-7 \quad (1) \end{aligned}$$

Using these two equations we can now use elimination to solve for x and y.

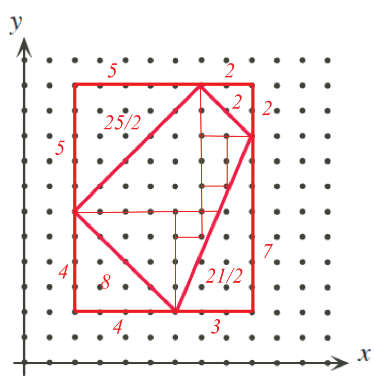
$$\begin{array}{ll} (2)\times 2 & 6x-4y+18=0 \\ (1)\times 3 & -(6x+9y-21=0) \\ \hline & -13y+39=0 \\ & y=3 \end{array}$$

$$0 = 2x+9-7$$
$$0 = 2x+2$$
$$x = -1$$

Therefore, our desired point is $(-1, 3)$

5. 2015-16 Problem of the Week (POTW)- Level C- Week 10
The dots on the diagram are one unit apart, horizontally and vertically. Determine the area of the figure.

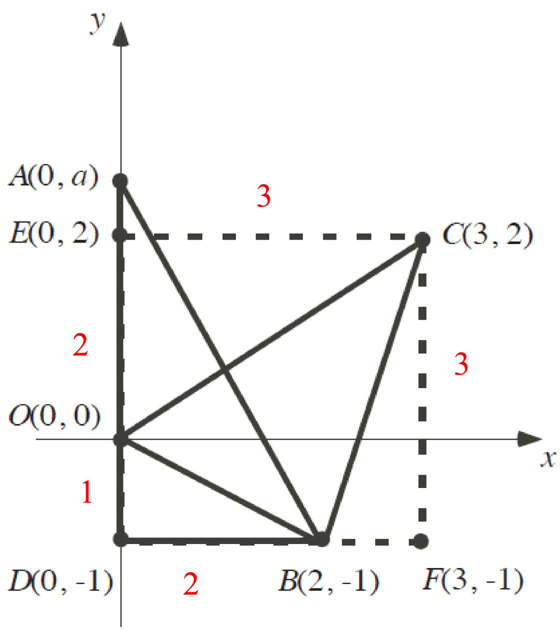
One method for determining the area is by creating a rectangle around the figure and then subtracting away the areas of the surrounding rectangles.



$$\begin{aligned} A &= 7\times 9 - \left(\frac{25}{2}+\frac{21}{2}+8+2\right) \\ &= 63-(23+10) \\ &= 63-33 \\ &= 30 \end{aligned}$$

Therefore the area of the figure is 30 units squared, whatever those units may be.

6. 2015-16 POTW- Level D- Week 10
In the diagram, $A(0,a)$ lies on the y-axis above the origin. If $\triangle ABD$ and $\triangle COB$ have the same area, determine the value of a.



We can find the area of $\triangle COB$ by using the same approach shown in question 6 and since $\triangle ADB$ is a right triangle finding the area can be done using the formula.

$$\begin{aligned} A_{COB} &= 3^2 - \frac{1}{2}(2\times 3 + 3\times 1 + 1\times 2) \\ &= 9 - \frac{1}{2}(6+3+2) \\ &= 9 - \frac{1}{2}(11) \\ &= \frac{18}{2} - \frac{11}{2} \\ &= \frac{7}{2} \end{aligned}$$

$$\begin{aligned} A_{ADB} &= \frac{1}{2}(2)(a+1) \\ A_{COB} &= a+1 \\ \frac{7}{2} &= a+1 \\ a &= \frac{5}{2} \end{aligned}$$

Therefore the have of a is $\frac{5}{2}$.

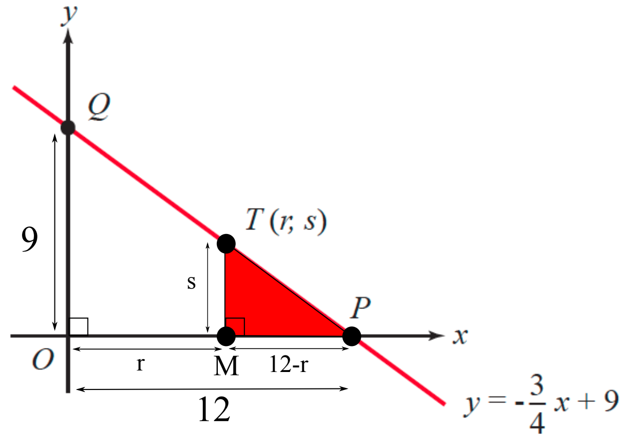
7. 2013-14 POTW- Level D

The line $y = -\frac{3}{4}x + 9$ crosses the x-axis at P and the y-axis at Q.

Point $T(r, s)$ lies on the line segment PQ.

The area of $\triangle POQ$ is three times the area of $\triangle TOP$.

Deterime the values of r and s, the coordinates of T.



We are given that $A_{POQ} = 3A_{TOP}$

$$\begin{aligned} A_{POQ} &= \frac{1}{2}(9)(12) \\ 54 &= 3\frac{1}{2}(12)(s) \\ \frac{54}{18} &= \frac{18s}{18} \\ s &= 3 \end{aligned}$$

We are given that $y = -\frac{3}{4}x + 9$ and Q is where the line crosses the y-axis. Thus, Q is the y-intercept and it's location is $Q(0, 9)$. To find P (the x-intercept) we can set $y = 0$ and solve for x.

$$\begin{aligned} y &= -\frac{3}{4}x + 9 \\ 0 &= -\frac{3}{4}x + 9 \\ -9 &= -\frac{3}{4}x \\ 3 &= \frac{1}{4}x \\ 12 &= x \\ x &= 12 \end{aligned}$$

Now we know $P(12, 0)$ which tells us the dimensions of our right triangle formed by the line, the x-axis and y- axis. By dropping a perpendicular line from T to the x-axis and calling the intersection M we form a similar triangle to $\triangle QOP$. We can find the dimensions of $\triangle TMP$ in terms of r and s.

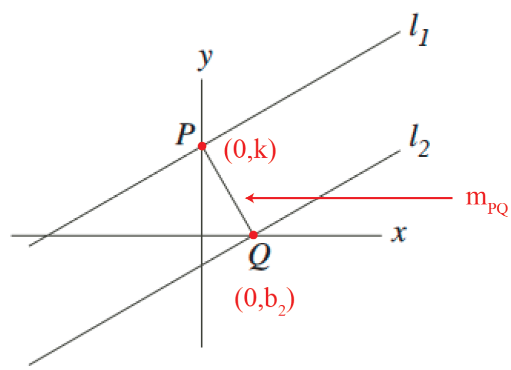
$$\begin{aligned}\frac{QO}{OP} &= \frac{TM}{MP}\\ \frac{9}{12} &= \frac{3}{12-r}\\ 12-r &= 4\\ r &= 8\end{aligned}$$

Therefore the coordinates of T are $(8,3)$.

8. 2010-11 POTW- Level D

Line l_1 has equation $y = mx+k$. Line l_1 cross the y-axis at point P and l_2 crosses the x-axis at the point Q. PQ is perpendicular to both line l_1 and line l_2.

Determine the y-intercept of l_2 in terms of m and k.



We are given the following

$$\begin{aligned}&l_1 : y = mx + k \implies P(0,k) \text{ is the y-intercept}\\ &m_{PQ} = -\frac{1}{m}\end{aligned}$$

Therefore we know $\mathscr{L}_{PQ} : y = -\frac{1}{m}x + k$.

Now to find Q. We know Q crosses the x-axis and is on $\mathscr{L}_{PQ}$. Thus be setting $x =$ in our equation for $\mathscr{L}_{PQ}$ we can find Q's coordinates.

$$\begin{aligned}0 &= \frac{1}{m}x + k\\ \frac{x}{m} &= k\\ x &= km \implies Q(km,0)\end{aligned}$$

Now since $\mathscr{L}_{PQ}$ is perpendicular to both l_1 and l_2 we know the two lines must be parallel.

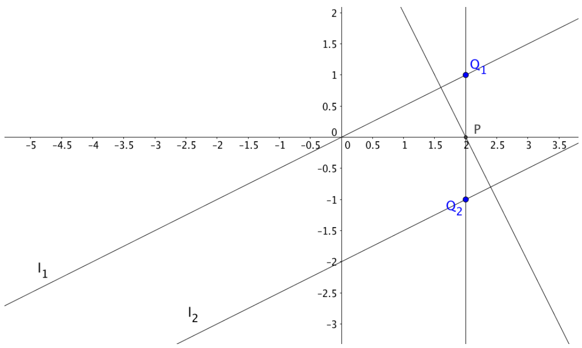
$$l_1 || l_2 \implies m_2 = m \text{ where } m_2 \text{ is the slope of } l_2$$

Now we have the slope of l_2 and a point, $Q(km,0)$ on the line. We can use these to find the y-intercept of l_2.

$$\begin{aligned}y &= mx + b_2\\ 0 &= m(km) + b_2\\ b_2 &= -m^2k\end{aligned}$$

Therefore the y-intercept in terms of m and k is $(0, -m^2k)$

9. 2012 Canadian Team Mathematics Contest Individual Problems Q3
The line $x = 2$ intersects the lines $y = -2x + 4$ and $y = \frac{1}{2}x + b$ at points a distance of 1 unit from each other. What are the possible values of b?



Find the intersection of $y = -2x + 4$ and $x = 2$.

$$y = -2x + 4$$
$$y = -2(2) + 4$$
$$y = 0$$

Thus $P(2, 0)$.

Since the points are a distance of 1 unit from each other. There are two possible lines $y = \frac{1}{2}x + b$, call them l_1 and l_2 and therefore two possible values for b. Let Q_1 and Q_2 be our two intersections with l_1 and l_2.

l_1 :$Q_1(2, 1)$

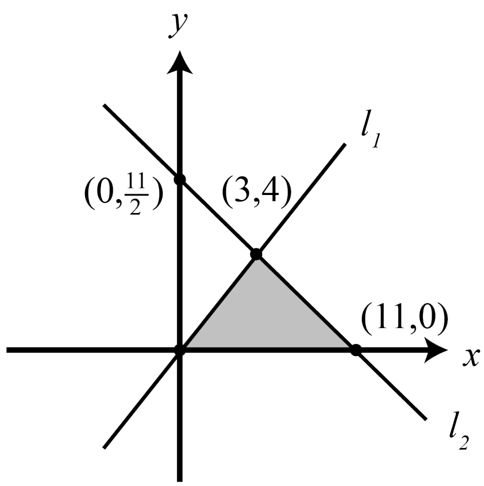
$$y = \frac{1}{2}x + b_1$$
$$1 = \frac{1}{2}(2) + b_1$$
$$b_1 = 0$$

l_2 :$Q_2(2, -1)$

$$y = \frac{1}{2}x + b_2$$
$$-1 = \frac{1}{2}(2) + b_2$$
$$b_2 = -2$$

Therefore the possible values for b are 0 and -2.

10. 2013 Canadian Team Mathematics Contest Individual Problems Q3
Find the area of the triangle formed by the lines $4x - 3y = 0$, $x + 2y = 11$ and the x-axis.



We need to find the point of intersection amongst the three lines.

$$4x - 3y = 0 \quad (1)$$
$$y = \frac{4}{3}x$$
$$x + 2y = 11 \quad (2)$$

$$(1) - 4(2) \quad \begin{array}{r} 4x - 3y = 0 \\ -(4x + 8y = 44) \\ \hline -11y = -44 \end{array}$$

$$y = 4 \text{ sub into (1)}$$
$$4x - 3(4) = 0$$
$$4x = 12$$
$$x = 3$$

Therefore the point of intersection between the two lines other than the x-axis is $(3, 4)$.

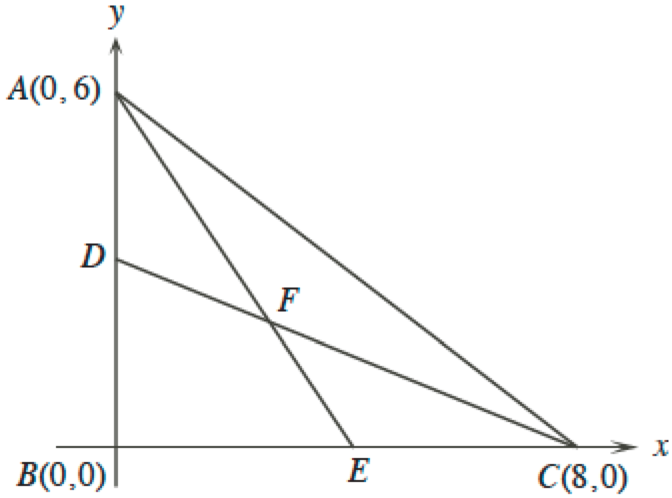
The x-intercepts of the other two lines are the other two points of intersection and based of the questions we can deduce they are $(0, 0)$ and $(11, 0)$.
Knowing these points gives us the height of the triangle, $h = 4$ and the base $b = 11$.

$$\begin{aligned} A &= \left(\frac{1}{2}\right)(4)(11) \\ &= 22 \end{aligned}$$

Therefore the area of the shaded triangle formed by the lines is 22 units squared.

11. 2011 Hypatia Q1

In the diagram, D and E are the midpoints of AB and BC respectively.



(a) Determine an equation of the line passing through the points C and D.

(b) Determine the coordinates of F, the point of intersection of AE and CD.

(c) Determine the area of $\triangle DBC$.

(d) Determine the area of quadrilateral $DBEF$.

(a) Points D$(0,3)$ and C$(8,0)$ allow quickly determine the y-intercept, point D, and slope of the line $m = -\frac{3}{8}$. Therefore the equation of the line is $y = -\frac{3}{8}x + 3 \quad (1)$.

(b) We know E is the midpoint of $B(0,0)$ and $C(8,0)$. Thus $E(4,0)$. From there we can quickly determine, as we did in (a), the slope and y-intercept for $\mathscr{L}_{AE}$. Thus the equation for $\mathscr{L}_{AE}$ is $y = -\frac{3}{2}x + 6$ (2).

Now by substitution we can find the intersection of the two lines, point F.

$$\begin{aligned} (1) \to (2) \qquad 8 \times (-\frac{3}{8}x + 3) &= (-\frac{3}{2}x + 6) \times 8 \\ -3x + 24 &= -12x + 48 \\ 9x &= 8 \\ x &= \frac{8}{3} \quad \text{sub into (1)} \\ y &= -\frac{3}{8}\left(\frac{8}{3}\right) + 3 \\ y &= -1 + 3 = 2 \\ F(\frac{8}{3}, 2) \end{aligned}$$

(c) We can find the midpoint of $B(0,0)$ and $A(0,6)$ to be $D(3,0)$. Since $\triangle DBC$ is a right triangle we know $A_{DBC} = \frac{1}{2}(8)(3) = 12$.

(d) Looking at the diagram we can find the area of the quadrilateral by taking the difference of $\triangle DBC$ and $\triangle FEC$.

$$\begin{aligned} A_{DBEF} &= A_{DBC} - A_{FEC} \\ &= 12 - \frac{1}{2}(4)(2) \\ A_{DBEF} &= 8 \end{aligned}$$

12. Find the length of the following line segments between points $O(0,0,0)$, $A(-6,1,0)$, and $B(0,3,-4)$.

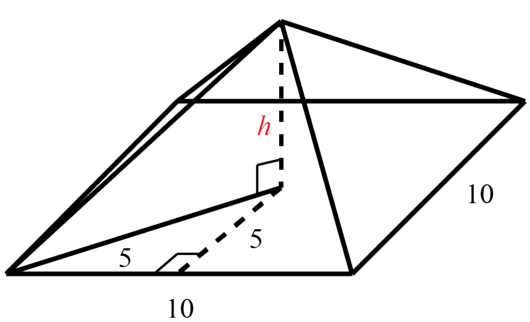
(a) OA

$$\begin{aligned} d_{OA} &= \sqrt{(-6)^2 + 1^2 + 0^2} \\ &= \sqrt{36+1+0} \\ &= \sqrt{37} \end{aligned}$$

(b) BA

$$\begin{aligned} d_{BA} &= \sqrt{(0-(-6))^2 + (3-1)^2 + (0-(-4))^2} \\ &= \sqrt{36+4+16} \\ &= \sqrt{56} \\ &= 2\sqrt{14} \end{aligned}$$

13. A square-based pyramid has a height of $\sqrt{31}$m and a base area of 100m^2. What is the length of the slant of the pyramid (length from the corner of the base to the top of the pyramid)?



We are given $h = \sqrt{31}$ m.

$$\begin{aligned} s &= \sqrt{5^2 + 5^2 + h^2} \\ s &= \sqrt{5^2 + 5^2 + 31} \\ s &= \sqrt{81} \\ s &= 9 \end{aligned}$$

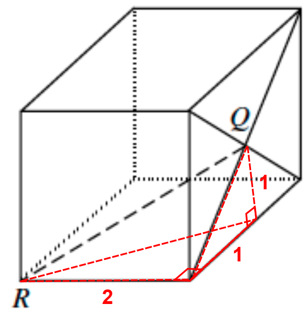
Therefore the length of the pyramids slant is 9 m.

14. 1998 Pascal Q21

Q is the point of intersection of the diagonals of one face of a cube whose edges have length 2 units. The length of QR is

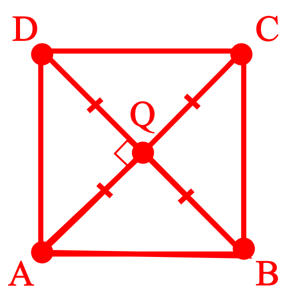
(A) 2 **(B)** $\sqrt{8}$ **(C)** $\sqrt{5}$

(D) $\sqrt{12}$ **(E)** $\sqrt{6}$

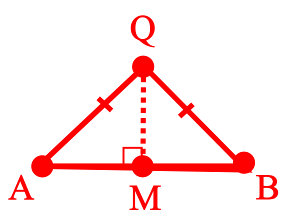


We know to use the Pythagorean Theorem for three-dimensions, but before we can apply the result we need to find the missing two lengths.

Consider the face of the square which Q is located. We know Q is the intersection of the diagonals of the square. We also know these diagonals of a square are perpendicular bisectors from a problem during the lecture. In other words, $\angle AQB = 90°$ and $AQ = QC = DQ = QB$. Now since $\triangle AQB$ is an isosceles triangle we know $\angle QAB = 45°$.



The pythagorean theorem in three-dimensions works with axes perpendicular to each other. Drop a perpendicular line from Q to AB.



Our next step is to find the lengths of AM and QM. Since $\triangle AQB$ is an isosceles triangle, then the perpendicular line to M is also a bisector of AB. Thus $AM = MB = 1$

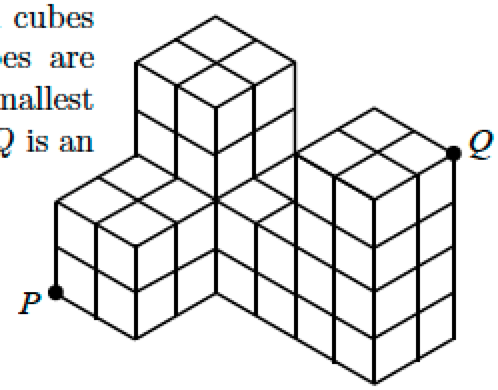
$$\angle QAB + \angle AMQ + \angle MQA = 180° \qquad \text{(Angle Sum Theorem-AST)}$$
$$45° + 90° + \angle MQA = 180°$$
$$\angle MQA = 45°$$

This tells us that $\triangle AMQ$ is also an isosceles triangle and thus $AM = MQ = 1$. Now we can finally apply the Pythagorean Theorem in three dimensions.

$$\begin{aligned} d_{RQ} &= \sqrt{2^2 + 1^2 + 1^2} \\ &= \sqrt{4 + 1 + 1} \\ &= \sqrt{6} \end{aligned}$$

15. 2014 Pascal Q23

In the diagram, the shape consists of 48 identical cubes with edge length $\sqrt{n}$. Entire faces of the cubes are attached to one another, as shown. What is the smallest positive integer n so that the distance from P to Q is an integer?



(A) 17 **(B)** 68 **(C)** 7
(D) 28 **(E)** 3

Place your x-,y-, and z-axes at point P.

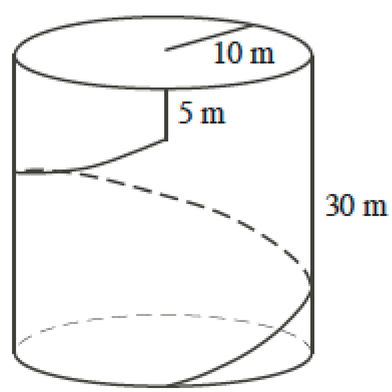
$$d_{PQ} = \sqrt{(-4\sqrt{n})^2 + (6\sqrt{n})^2 + (4\sqrt{n})^2}$$
$$= \sqrt{36n + 16n + 16n}$$
$$= \sqrt{68n}$$

Note that the value under the square root must be a perfect square for the distance from P to Q to be an integer.

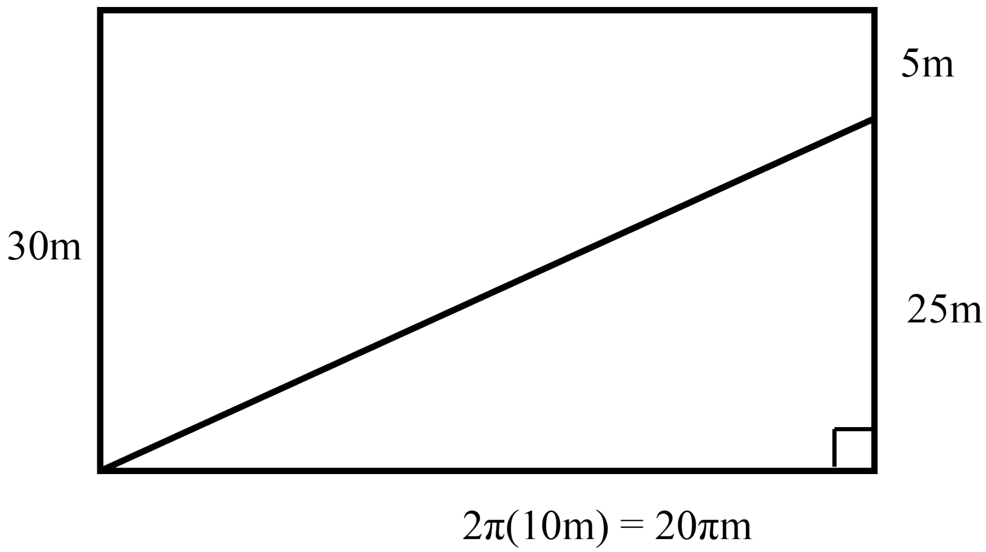
$$68 \times n = 2 \times 34 \times n$$
$$= 2 \times 2 \times 17 \times n$$
$$= 2^2 \times 17 \times n$$
$$n = 17$$

16. 2013 Pascal Q22

A water tower in the shape of a cylinder has radius 10 m and height 30 m. A spiral staircase, with constant slope, circles once around the outside of the water tower. A vertical ladder of height 5 m then extends to the top of the tower. Which of the following is closest to the total distance along the staircase and up the ladder to the top of the tower?



(A) 72.6 m **(B)** 320.2 m **(C)** 74.6 m
(D) 67.6 m **(E)** 45.1 m



When we unravel the cylinder's sides we get a rectangle and can apply the Pythagorean Theorem.

$$\begin{aligned} d &= \sqrt{(20\pi)^2 + (25)^2} \\ &= 5\sqrt{(4\pi)^2 + 5^2} \\ &= 5\sqrt{16\pi^2 + 25} \end{aligned}$$