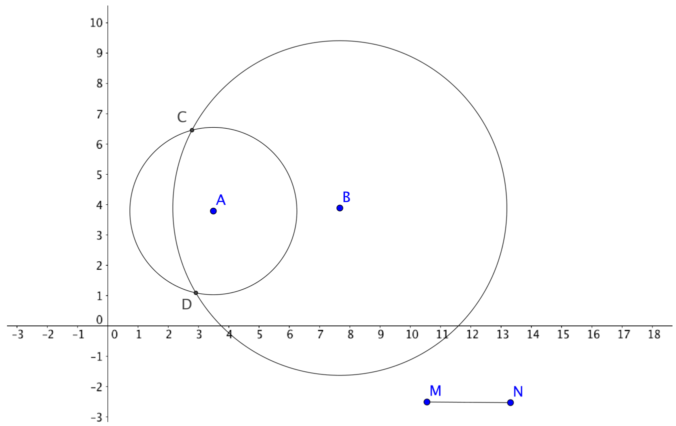
Intermediate Math Circles
Wednesday, March 29, 2017
Problem Set 7

1. Using GeoGebra(*geogebra.org*), determine the locus of points that are twice as far from point A as they are from point B.



Steps:

i Construct and label two points A and B.

ii Construct a line segment of arbitrary length. Label the end points M and N.

iii Construct a circle with centre A and radius MN.
Note: Can do this using the *Input:* bar and the command
Circle[<Point>,<Radius Number>].

iv Construct a circle with centre B and radius twice the length of MN.

v Select the points of intersection of the two circles and label them C and D.
Note: You may need to adjust the length of line segment MN so that the circles intersect.

vi Right click on points C and D and select *Trace On*.

vii Vary the length of line segment MN.

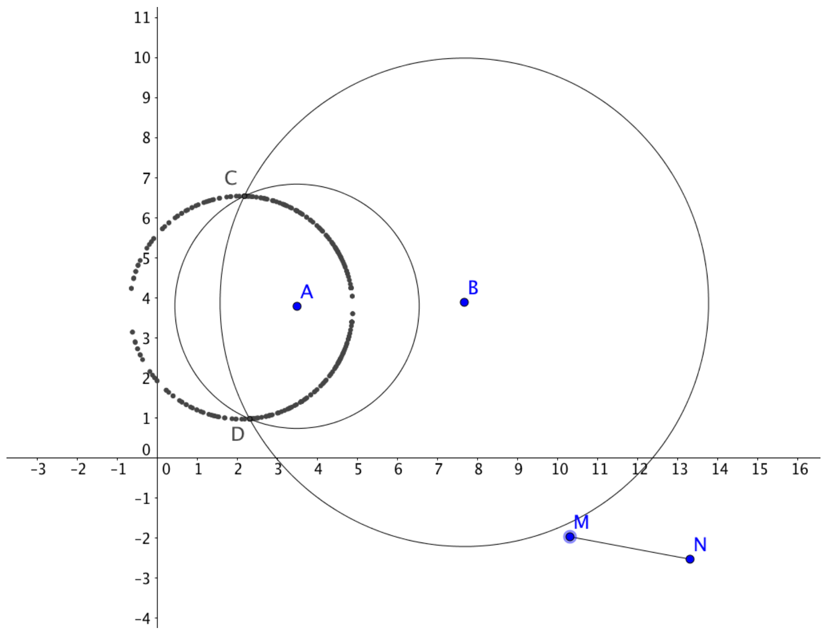
Questions:

(a) Describe the locus

(b) Change the location of point A. Describe how the locus changes

i. when points A and B are closer together

ii. when points A and B are farther apart

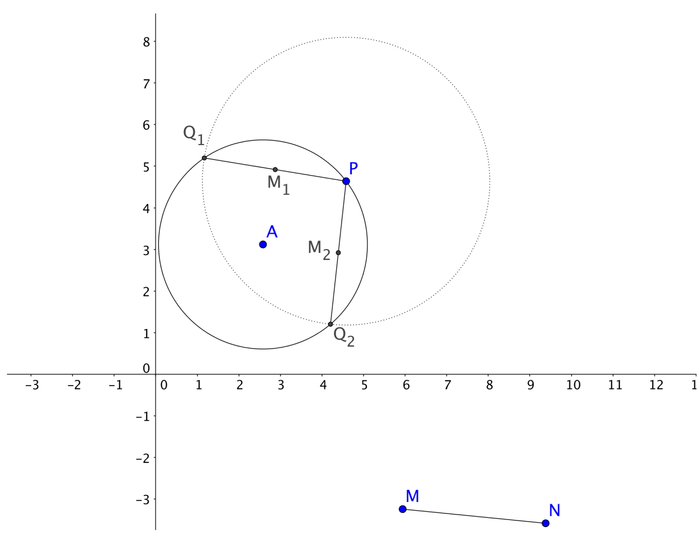


(a) It's a circle who's centre is on the line through A and B.

(b) i. The circle's radius decreases.

ii. The circle's radius increases.

2. Using GeoGebra(*geogebra.org*), consider chords of equal length drawn in a circle. Determine the locus of the midpoints of the chords.



Steps:

i Construct a line segment MN. This will be the length of the chord.

ii Construct a circle with centre A and point P.
Hint: the command **Circle**[**<Point>**,**<Point>**] will be helpful

iii Construct a circle with centre P and radius of length MN.
Hint: remember command **Circle**[**<Point>**,**<Radius Number>**]

iv Call the intersections of your two circles Q_1 and Q_2.
Note: You can hide your recently created circle by right clicking on the circle and unselecting *Show Object* and *Show Label*.

v Using the line segment command create cords PQ_1 and PQ_2.

vi Construct the midpoints of line segments PQ_1 and PQ_2. Rename the midpoints M_1 and M_2.

vii Right click on points M_1 and M_2 and select *Trace On*.

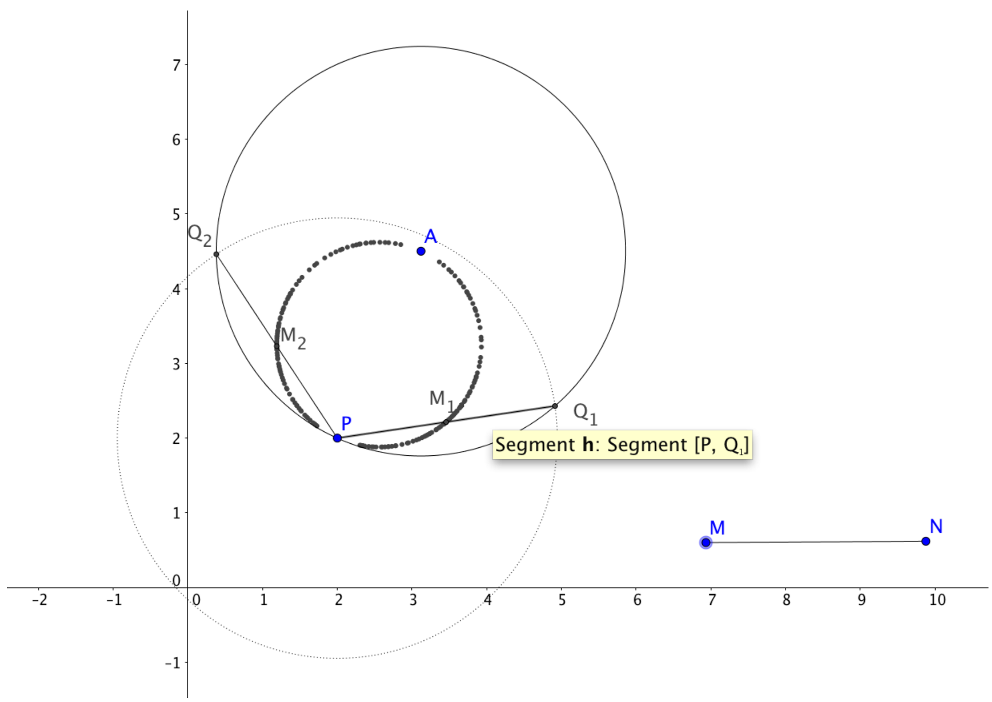
viii Vary the length of line segment MN.

Questions:

(a) Describe the locus of midpoints of the chords

(b) Where is do you suspect the centre of the locus is located?

(c) How would the locus change if you only had one of M_1 and M_2?

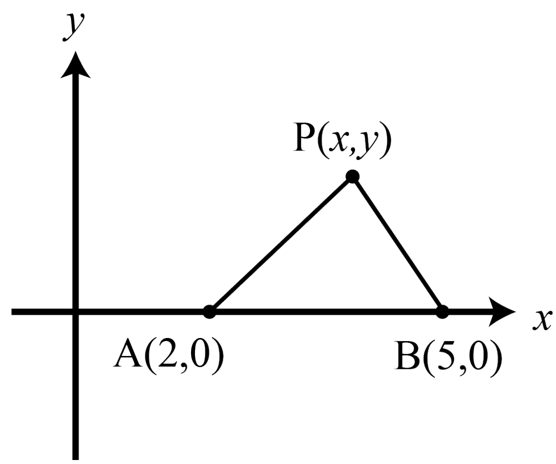


(a) It is a circle and points A and P are diametrically opposite points on the circle.

(b) The midpoint of A and P.

(c) You would only have a semicircle.

3. Given the points $A(2,0)$ and $B(5,0)$, find the equation of the locus of points that are twice as far from point A as they are from point B.



Let $P(x, y)$ be a point on our locus.

$$\begin{aligned} \text{Given: } AP &= 2(PB) \\ \left(\sqrt{(x-2)^2+y^2}\right)^2 &= \left(2\sqrt{(5-x)^2+y^2}\right)^2 \quad \text{Square Both Sides- SBS} \\ (x^2-4x+4)+y^2 &= 4[(25-10x+x^2)+y^2] \\ x^2-4x+4+y^2 &= 100-40x+4x^2+4y^2 \\ 0 &= 3x^2-36x+3y^2+96 \\ 0 &= x^2-12x+y^2+32 \end{aligned}$$

4. Determine an equation for each for the following circles

(a) centre $(0, 0)$, through $(-2, 3)$

$$\begin{aligned} x^2+y^2 &= r^2 \\ (-2)^2+3^2 &= r^2 \\ 4+9 &= r^2 \\ r &= \sqrt{13} \\ \implies \quad x^2+y^2 &= 13 \end{aligned}$$

(b) centre $(0, 0)$, x-intercepts at ± 8

Given $(0, 0)$ is the centre and both points are 8 units from the centre, we know $r = 8 \implies x^2 + y^2 = 64$

(c) centre $(3, 4)$, through $(0, 0)$

Centre is $(h, k) = (3, 4)$ and point $(x, y) = (0, 0)$ is on the circle.

$$\begin{aligned} (x-h)^2+(y-k)^2 &= r^2 \\ (0-3)^2+(0-4)^2 &= r^2 \\ r^2 &= 25 \\ r &= 5 \\ \implies \quad (x-3)^2+(y-4)^2 &= 25 \end{aligned}$$

(d) centre $(-1, 3)$, through $(1, -1)$

Centre is $(h, k) = (-1, 3)$ and point on a circle is $(x, y) = (1, -1)$.

$$\begin{aligned}(x-h)^2 + (y-k)^2 &= r^2\\ [1-(-1)]^2 + [-1-3]^2 &= r^2\\ 2^2 + (-4)^2 &= r^2\\ 4 + 16 &= r^2\\ r^2 &= 20\\ \implies (x+1)^2 + (y-3)^2 &= 20\end{aligned}$$

(e) centre $(-2, -2)$, y-intercept -2

Since the centre $(-2, -2)$ and y=intercept $(0, -2)$ lie on the same horizontal line $y = -2$ and are two units apart, we know the radius, $r = 2$.

$(x+2)^2 + (y+2)^2 = 4$

5. (a) Show that the points $P(-2, 4)$ and $Q(2, -4)$ are both on the circle $x^2 + y^2 = 20$.

For any point (a, b) to be on the circle, the values a and b must satisfy the equation $a^2 + b^2 = 20$.

Check for points $(-2, 4)$ and $(2, -4)$.
$(-2)^2 + 4^2 = 4 + 16 = 20$ as required.
$(2)^2 + (-4)^2 = 4 + 16 = 20$ as required.

(b) Show that PQ is a diameter of the circle

PQ is a diameter if its midpoint is the centre of the circle.

$\left(\frac{-2+2}{2}, \frac{4+(-4)}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$

6. Determine the equations of the circles with the given diameters

(a) from $(-3, 5)$ to $(3, -5)$

$\left(\frac{-3+3}{2}, \frac{5+(-5)}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$

Therefore, $(0, 0)$ is the circle's centre.

(b) from $(-1, 2)$ to $(5, 8)$

$\left(\frac{-1+5}{2}, \frac{2+8}{2}\right) = \left(\frac{4}{2}, \frac{10}{2}\right) = (2, 5)$

Therefore, $(2, 5)$ is the circle's centre.

7. For the circle given by $x^2 + y^2 = 34$,

(a) show that the line segment from $P(-5, 3)$ to $Q(3, 5)$ is a chord of the circle;

To be a cord, both P and Q need to be points on the circle.
Check P: $(-5)^2 + (3)^2 = 25 + 9 = 34$
Check Q: $3^2 + 5^2 = 9 + 25 = 34$

(b) find the midpoint M of the chord;

$\left(\frac{-5+3}{2}, \frac{3+5}{2}\right) = \left(-\frac{2}{2}, \frac{8}{2}\right) = (-1, 4)$

Therefore, midpoint of the chord is $(-1, 4)$.

(c) show that $MO \perp PQ$

To show $MO \perp PQ$, we need to show that $m_{MO} = -\frac{1}{m_{PQ}}$

The equation of the circle ,$x^2+y^2 = 34$, tells us that $O(0,0)$ is the centre of the circle.

$m_{MO} = \frac{0-4}{0-(-1)} = -4$

$m_{PQ} = \frac{5-3}{3-(-5)} = \frac{2}{8} = \frac{1}{4}$

Since $-\frac{1}{m_{MO}} = -\frac{1}{(-4)} = \frac{1}{4} = m_{PQ}$

Therefore we know $MO \perp PQ$.

8. A circle passes through the points $A(-1,1)$ and $B(6,0)$ and has its centre on the line $x + 3y + 7 = 0$. Find the equation of the circle.

Let (h,k) be the centre of the circle. Given that (h,k) is on line $x + 3y + 7 = 0$.

$$\text{i.e.} \quad h + 3k + 7 = 0$$
$$h = -3k - 7$$

$A(-1,1)$ and $B(6,0)$ are on the circle.

$$\textbf{Recall:} \quad (x-h)^2 + (y-k)^2 = r^2$$

$$(-1-h)^2 + (1-k)^2 = r^2$$
$$(1+h)^2 + (1-k)^2 = r^2 \quad (1)$$

$$(6-h)^2 + (0-k)^2 = r^2$$
$$(6-h)^2 + k^2 = r^2 \quad (2)$$

$$\textbf{Set (1) = (2)} \quad (1+h)^2 + (1-k)^2 = (6-h)^2 + k^2$$
$$1 + 2h + h^2 + 1 - 2k + k^2 = 36 - 12h + h^2 + k^2$$
$$2 + 2h - 2k = 36 - 12h$$

$$\textbf{Sub in} \quad h = -3k - 7$$
$$2 + 2(-3k-7) - 2k = 36 - 12(-3k-7)$$
$$2 - 6k - 14 - 2k = 36 + 36k + 84$$
$$-12 - 8k = 120 + 36k$$
$$\frac{-44k}{-44} = \frac{132}{-44}$$
$$k = -3$$
$$h = -3(-3) - 7$$
$$= 2$$

Thus, the circle's centre is located at $(2,-3)$.

Now, to find the radius using the centre $(2, -3)$ and point $A(6, 0)$.

$$\begin{aligned} (x-h)^2 + (y-k)^2 &= r^2 \\ (6-2)^2 + (0-(-3))^2 &= r^2 \\ 4^2 + 3^2 &= r^2 \\ r^2 &= 25 \\ r &= 5 \end{aligned}$$

Therefore the circles radius is 5.