Intermediate Math Circles
Wednesday, March 8, 2017
Problem Set 5

1. What is the smallest positive integer x for which

$$\sum_{i=1}^{100} ix$$

is a perfect square?

Solution: We begin by simplifying the sum.

$$\begin{aligned}\sum_{i=1}^{100} ix &= x + 2x + 3x + \cdots + 100x \\ &= x(1 + 2 + 3 + \cdots 100) \\ &= x\left(\frac{100(101)}{2}\right) \\ &= 5050x\end{aligned}$$

The prime factorization of 5050 is $2 \cdot 5^2 \cdot 101$. For $5050x$ to be a perfect square, its prime factors must come in pairs. Therefore, the smallest value of x such that the prime factors of $5050x$ come in pairs is $x = 2(101) = 202$.

2. Consider a sequence where $t_k = 3^k - 2k + 2$. Calculate $\sum_{k=1}^{n} t_k$.

Solution: Each term t_k can be thought of as the sum of 3^k and $-2k + 2$. Therefore,

$$\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} 3^k + \sum_{k=1}^{n}(-2k + 2).$$

The first of these two sums is a geometric series with n terms, the first term is 3 and the common ratio is 3. Therefore this sum is $\frac{3(1-3^n)}{1-3}$ which simplifies to $\frac{3}{2}(3^n - 1)$.

The second sum is an arithmetic series with n terms, the first term is 0 and the common difference is -2. Therefore this sum is $\frac{n}{2}\left[2(0) + (n-1)(-2)\right]$ which simplifies to $-n(n-1)$.

Therefore, $\sum_{k=1}^{n} t_k = \frac{3}{2}(3^n - 1) - n(n-1)$.

3. In a geometric sequence, the first term is 7, the last term is 448, and the sum is 889. Find the third term.

   Solution:

   Let the number of terms in the sequence be n and let the common ratio be r. The last term of the sequence is $t_n = 7r^{n-1} = 448$. We multiply both sides of this equation by r to obtain $7r^n = 448r$.

   The sum of the terms in the sequence is $7\left(\frac{1-r^n}{1-r}\right) = 889$. Therefore $889-889r = 7-7r^n$. We substitute $7r^n = 448r$ to obtain $889 - 889r = 7 - 448r$. Solving this equation we get $r = 2$.

   Therefore $t_3 = 7(2^2) = 28$.

4. The sum of the first n terms of a sequence is $n(n+1)(n+2)$. What is the 10th term of the sequence?

   Solution:

   Let S_n be the sum of the first n terms. Therefore,

$$\begin{aligned} t_{10} &= S_{10} - S_9 \\ &= 10(11)(12) - 9(10)(11) \\ &= 330 \end{aligned}$$

5. Evaluate the sum $\sum_{i=1}^{28}\left[\frac{1}{i} - \frac{1}{i+2}\right]$

   Solution:

$$\begin{aligned} &\sum_{i=1}^{28}\left[\frac{1}{i} - \frac{1}{i+2}\right] \\ &= \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{26} - \frac{1}{28} + \frac{1}{27} - \frac{1}{29} + \frac{1}{28} - \frac{1}{30} \\ &= 1 + \frac{1}{2} - \frac{1}{29} - \frac{1}{30} \\ &= \frac{623}{435} \end{aligned}$$

6. Find $9 + 99 + 999 + 9999 + \ldots$ to n terms.

   Solution:

   The sum is equivalent to $(10-1) + (10^2-1) + (10^3-1) + \cdots + (10^n-1)$. Rearranging we obtain $10+10^2+10^3+\cdots+10^n-n$. The first n terms of this sum is a geometric series where the first term is 10 and the common ratio is 10. Therefore, the sum is $10\left(\frac{1-10^n}{1-10}\right) - n$ which simplifies to $\frac{10}{9}(10^n-1) - n$.