UNIVERSITY OF WATERLOO

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Centre for Education in
Mathematics and Computing

# Grade 6 Math Circles

April 4/5, 2017

## *Math Jeopardy*

## Exponents and Primes

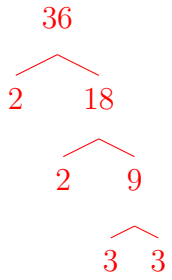
**$100** What is $8^7$?

$8^7 = 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 = 2097152$

**$200** What is the 11th prime number?

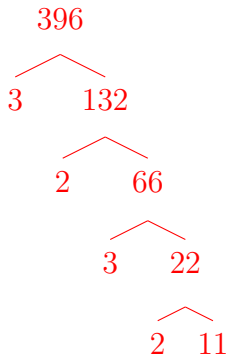
The prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,... so the 11th prime number is 31.

**$300** Find all the prime factors of 36.




Thus, the prime factors are 2 and 3.

**$400** What is the prime factorization of 396?




$396 = 2^2 \times 3^2 \times 11$

**$500** A palindromic prime is a prime number that is read the same forwards and backwards. What is the smallest 3-digit palindromic prime?

101

## Sorting

**$100** Sort these alphabetically: Regulus, Scorpius, Remus, Kingsley, Sirius

Kingsley, Regulus, Remus, Scorpius, Sirius

**$200** What sorting algorithm uses the "divide and conquer" strategy?

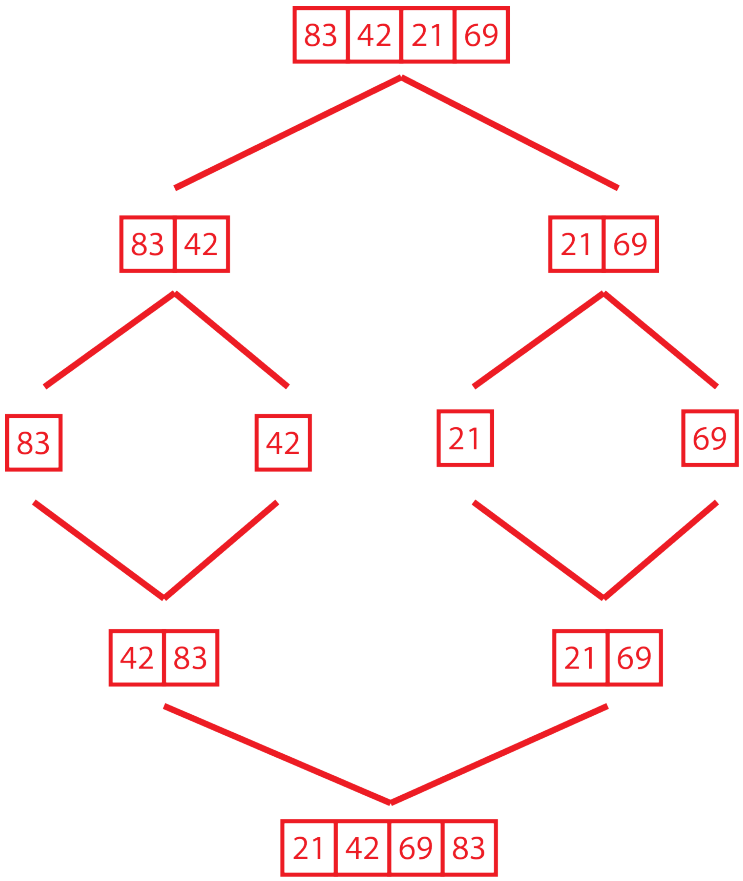
Merge sort.

**$300** How many steps does it take to sort 6, 4, 1, 7 using Insertion sort?

| Unsorted | Sorted |
|---|---|
| 6 4 1 7 | |
| 4 1 7 | **6** |
| 1 7 | **4** 6 |
| 7 | **1** 4 6 |
| | **7** 1 4 6 |
| | 1 **7** 4 6 |
| | 1 4 **7** 6 |
| | 1 4 6 **7** |

Thus, 7 steps.

**$400** Draw a tree diagram to sort 83, 42, 21, 69




**$500** How many times do you need to go through every pair in the list 3, 9, 4, 2, 0 when sorting it with Bubble sort?

5 times.

## Patterns

**$100** What are the next 3 terms in this pattern: 3, 15, 27, 39, 51, ...

The rule of the pattern is to add 12 to get the next term. Thus, the next 3 terms are 63, 75, 87.

**$200** What are the next 3 terms in this pattern: 4, 16, 64, 256, 1024, ...

The rule of the pattern is to multiply by 4 to get the next term. Thus, the next 3 terms are 4096, 16 384, 65 536.

**$300** What is the sum of the whole numbers from 1 to 572?

$$\begin{aligned} 572 \times (572 + 1) \div 2 &= 572 \times 573 \div 2 \\ &= 327756 \div 2 \\ &= 163878 \end{aligned}$$

**$400** What is the 16th term of the Fibonacci sequence?

The first 16 terms of the Fibonacci sequence are: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

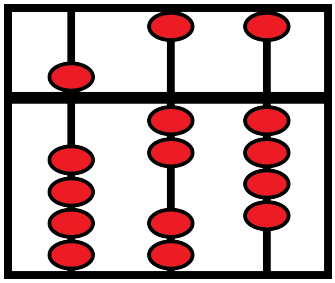
Thus, the 16th term of the Fibonacci sequence is 987.

**$500** What is the next term in this pattern: 8, 22, 64, 190, 568, ...

The rule is to multiply by 3 and then subtract by 2 to get the next term. Thus, $568 \times 3 - 2 = 1702$

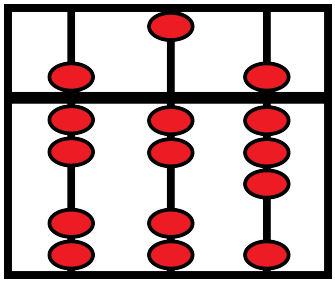
## Abacus

**$100** What is this number?

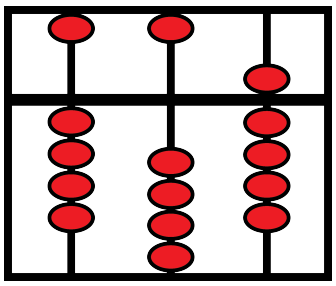


524

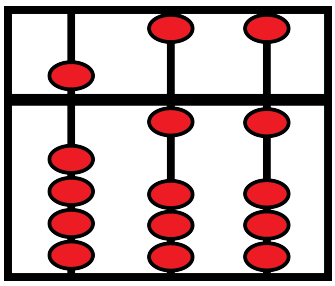
**$200** How do you show 728 on the abacus?



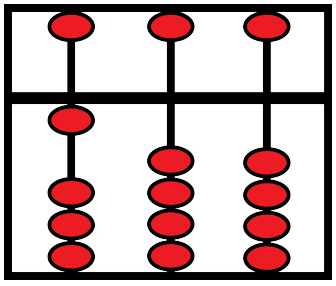
**$300** What will this abacus look like after you add 102?



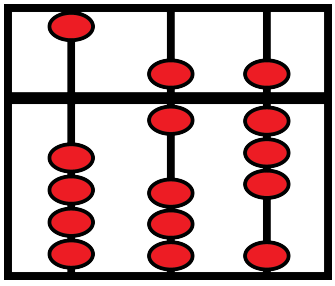
Solution:



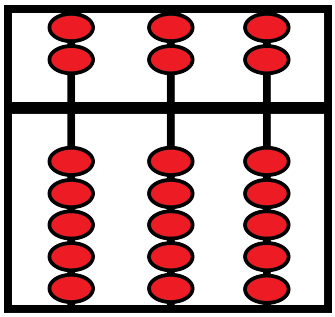
**$400** What will this abacus look like after you subtract 32?



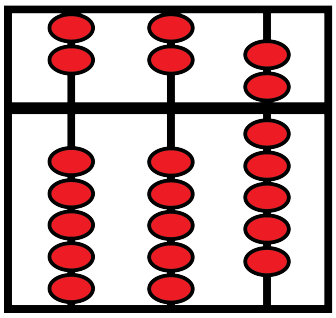
Solution:

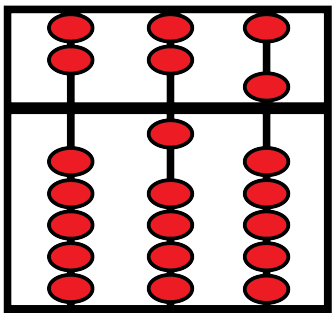


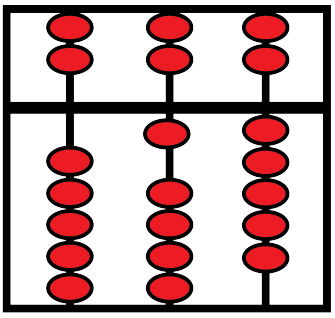
**$500** Using the Chinese abacus, (with 2 beads in the uppder deck, 5 beads in the lower deck), show 15 in 3 different ways.



Solution:







## Counting

**$100** How many ways can you make a PBJ sandwich with these ingredients? You need to use 1 type of bread, 1 type of jam, and 1 type of peanut butter.

- **Bread:** White, whole wheat, 9 grain
- **Jam:** Grape, strawberry, raspberry, apricot, blackberry
- **Peanut butter:** Crunchy, smooth

Using Fundamental Counting Principle, $3 \times 5 \times 2 = 30$ ways to make a PBJ.

**$200** A 5-character password contains first 3 digits and then 2 letters. How many different passwords are possible if repeats are allowed?

Using Fundamental Counting Principle, $10 \times 10 \times 10 \times 26 \times 26 = 10^3 \times 26^2 = 676000$

**$300** How many ways can you shuffle 9 cards?

$9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362880$

**$400** How many ways are there to choose 2 flavours of frozen yogurt to combine if there are 101 flavours total to choose from?

$${}_{101}C_2 = \frac{101!}{2!(101-2)!} = \frac{101!}{2!99!} = \frac{101 \times 100}{2 \times 1} = \frac{10100}{2} = 5050$$

**$500** The top 3 contestants in the math contest will win gold, silver, and bronze medals. How many ways can the medals be given out if there are 64 contestants?

$${}_{64}P_3 = \frac{64!}{(64-3)!} = \frac{64!}{61!} = 64 \times 63 \times 62 = 249984$$

## Clock Arithmetic

**$100** Write 3 pm mathematically (in mod 12).

3 pm o'clock can also be written as 15 o'clock (on a 24 hour clock).

Thus, $15 \equiv 3 \bmod 12$ since $15 \div 12 = 1$ Remainder 2

**$200** If it is 85 hours after midnight, what time is it?

We use mod 24 since we want to know whether the time is in the morning or afternoon:

$85 \div 24 = 3$ Remainder $13 \rightarrow 85 \equiv 13 \bmod 24$. So, it will be 13 o'clock or 1pm.

Or, we use mod 12: $85 \div 12 = 7$ Remainder $1 \rightarrow 85 \equiv 1 \bmod 12$. So it will be 1 o'clock but this does not tell us what time of day it is.

**$300** What is $4 + 7 \bmod 9$?

$4 + 7 = 11 \equiv 2 \bmod 9$

**$400** What is $3 \times (5 + 9) \bmod 20$?

$3 \times (5 + 9) = 3 \times 14 = 42 \equiv 2 \bmod 20$

**$500** Hermione has knit 592 scarves, Ron has knit 15, and Harry has knit 43. They plan to send packs of 27 scarves. How many scarves will be left over?

$592 + 15 + 43 = 650 \equiv 2 \bmod 27$

Therefore, there will be 2 scarves left over.

## Magic and Latin Squares

**$200** What is $x$ in this magic square?

| | | |
|---|---|---|
| 6 | | 8 |
| | 5 | $x$ |
| | | 4 |

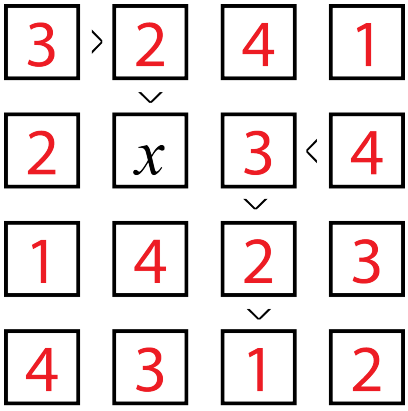
$6 + 5 + 4 = 15$ and so we have $8 + x + 4 = 15$ and thus $x = 15 - 8 - 4 = 3$

**$400** Find $x$ in this calcudoku puzzle.

| | | | |
|---|---|---|---|
| 2÷ 4 | 2 | 2÷ 1 | 7+ 3 |
| 6x 3 | 4÷ 1 | 2 | 4 |
| 2 | 4 | 3÷ 3 | 1 |
| 1 | 24x 3 | 4 | $x$ |

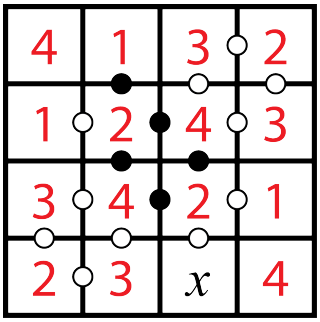
$x = 2$

**$600** Find $x$ in this futoshiki puzzle.



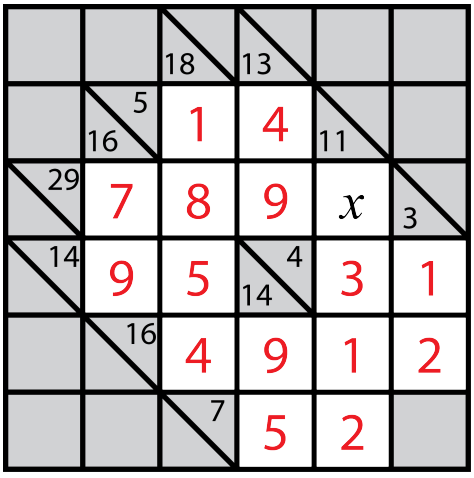

$x = 1$

**$800** Find $x$ in this kropki puzzle.




$x = 1$

**$1000** Find $x$ in this kakuro puzzle.




$x = 5$

# Probability

**$200** What is the probability of picking a face card (jack, queen, or king) from a deck of cards?

$\frac{12}{52} = \frac{3}{13} = 0.23 = 23\%$

**$400** What is the probability of flipping a coin 4 times and getting heads every time?

$$P(\text{heads}) = \frac{1}{2}$$

$$\begin{aligned} P(4 \text{ heads}) &= P(\text{heads} \cap \text{heads} \cap \text{heads} \cap \text{heads}) \\ &= P(\text{heads}) \times P(\text{heads}) \times P(\text{heads}) \times P(\text{heads}) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= (\frac{1}{2})^4 \\ &= 0.0625 = 6.25\% \end{aligned}$$

**$600** 2 white erasers and 3 pink erasers are in a pencil case. What is the probability that a pink eraser is picked out first, then a white eraser, then another pink eraser? (If you do not put erasers back in the pencil case in between picks.)

$$\begin{aligned} & P(\text{first pink} \cap \text{second white} \cap \text{third pink}) \\ =& P(\text{first pink}) \times P(\text{second white}) \times P(\text{third pink}) \\ =& \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} \\ =& \frac{12}{60} \\ =& 0.2 = 20\% \end{aligned}$$

**$800** What is the probability of picking a clubs card OR an even numbered card (face cards no not count)?

$P(\text{clubs})= \frac{12}{52}$ $P(\text{even})= \frac{20}{52}$ $P(\text{clubs AND even})= \frac{5}{52}$

$$\begin{aligned} P(\text{clubs OR even}) &= P(\text{clubs} \cup \text{even}) \\ &= P(\text{clubs}) + P(\text{even}) - P(\text{clubs AND even}) \\ &= \frac{12}{52} + \frac{20}{52} - \frac{5}{20} \\ &= \frac{28}{52} = 0.54 = 54\% \end{aligned}$$

**$1000** This Markov Chain shows the probability of which city a traveling circus will perform at next week. The circus only visits Hamilton, Kingston, and Waterloo.

$$\begin{array}{c} \\ H \\ K \\ W \end{array} \begin{array}{c} \begin{array}{ccc} H & K & W \end{array} \\ \begin{pmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{5}{7} & 0 & \frac{2}{7} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \end{array}$$

What is the probability that, starting in Hamilton, the circus will go to Kingston for 1 week, then Waterloo for 2 weeks.

$$\begin{aligned} P((H \to K) \cap (K \to W) \cap (W \to W)) &= P(H \to K) \times P(K \to W) \times P(W \to W) \\ &= \frac{3}{4} \times \frac{2}{7} \times \frac{1}{3} \\ &= \frac{6}{84} = 0.0714 = 7.14\% \end{aligned}$$

## Game Theory

**$200** Find the Nash Equilibrium(s) based on the following payoff table:

| | | Player 2 A | Player 2 B |
|---|---|---|---|
| Player 1 | A | 0, 0 | 3, 1 ★ |
| | B | ★ 3, 2 | ★ 4, 5 ★ |

Player 1 and 2 both choose B.

**$400** Find the Nash Equilibrium(s) based on the following payoff table:

| | | Player 2 A | Player 2 B |
|---|---|---|---|
| Player 1 | A | ★ 21, 9 ★ | 7, 6 |
| | B | 14, 10 | ★ 11, 13 ★ |

2 Nash Equilibrium(s), when both players choose the same move.

**$600** Find the Nash Equilibrium(s) based on the following payoff table:

| | | Player 2 A | Player 2 B |
|---|---|---|---|
| Player 1 | A | 2, 8 ★ | 7, 7 |
| | B | ★ 3, 2 ★ | ★ 10, 0 |

Player 1 chooses B and player 2 chooses A.

**$800** Find the Nash Equilibrium(s) based on the following payoff table:

| | | Player 2 A | Player 2 B | Player 2 C | Player 2 D |
|---|---|---|---|---|---|
| Player 1 | A | 1, 3 | ★ 4, 2 | 1, 4 ★ | 2, 2 |
| | B | 0, 2 ★ | 3, 2 ★ | 0, 2 ★ | 3, 1 |
| | C | 2, 2 | 2, 4 | 1, 5 ★ | ★ 4, 3 |
| | D | ★ 3, 0 | ★ 4, 2 | ★ 3, 3 ★ | ★ 4, 1 |

Player 1 chooses D and player 2 chooses C.

**$1000** Counting from 36 game follows these rules:

- Player A starts at 36.
- Each turn, a player can count down 1, 2, or 3 numbers.
- The player who says 0 wins.

What should Player A do to guarantee a win?

Using backwards induction, you can find that every multiple of 4 is a "winning number."

**0** 1 2 3 **4** 5 6 7 **8** 9
10 11 **12** 13 14 15 **16** 17 18 19
**20** 21 22 23 **24** 25 26 27 **28** 29
30 31 **32** 33 34 35 **36**

## Pascal's Triangle

**$200** What is the 6th row of Pascal's Triangle?

Row 0: 1
Row 1: 1 1
Row 2: 1 2 1
Row 3: 1 3 3 1
Row 4: 1 4 6 4 1
Row 5: 1 5 10 10 5 1
Row 6: 1 6 15 20 15 6 1

**$400** What is the 8th triangular number?

Row 0: 1
Row 1: 1 1
Row 2: 1 2 1
Row 3: 1 3 3 1
Row 4: 1 4 6 4 1
Row 5: 1 5 10 10 5 1
Row 6: 1 6 15 20 15 6 1
Row 7: 1 7 21 35 35 21 7 1
Row 8: 1 8 28 56 70 56 28 8 1
Row 9: 1 9 36 84 126 126 84 36 9 1
Row 10: 1 10 45 120 210 252 210 120 45 10 1

The 8th triangular number is 36.

You can also find $\frac{8 \times (8+1)}{2} = \frac{8 \times 9}{2} = \frac{72}{2} = 36$

**\$600** What is $1 + 5 + 10 + 5 + 1$ expressed as a power of 2?

I.e. What is $n$ for $2^n = 1 + 5 + 10 + 10 + 5 + 1$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Row 0: | | | 1 | | | |
| Row 1: | | 1 | | 1 | | |
| Row 2: | 1 | | 2 | | 1 | |
| Row 3: | | 1 | | 3 | | 3 | | 1 |
| Row 4: | 1 | | 4 | | 6 | | 4 | | 1 |
| Row 5: | 1 | 5 | 10 | 10 | 5 | 1 |

Thus, $2^5$ or $n = 5$.

**\$800** According to the hockey stick pattern, what are the 5 numbers that add to 35?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Row 0: | | | | 1 | | | |
| Row 1: | | | 1 | | 1 | | |
| Row 2: | | 1 | | 2 | | 1 | |
| Row 3: | | 1 | | 3 | | 3 | | 1 |
| Row 4: | 1 | | 4 | | 6 | | 4 | | 1 |
| Row 5: | 1 | 5 | 10 | 10 | 5 | 1 |
| Row 6: | 1 | 6 | 15 | 20 | 15 | 6 | 1 |
| Row 7: | 1 | 7 | 21 | 35 | 35 | 21 | 7 | 1 |

Thus, $35 = 1 + 3 + 6 + 10 + 15$

**\$1000** This is a section from Pascal's triangle. Find $x$.

3003 2002 1001 364

5005 3003 1365

8008 4368

$x$

$x = 8008 + 4365 = 12376$

## Number Systems and Bases

**$200** What is this Egyptian number?

1 012 081

**$400** What is this number: $(3)(10^6) + (9)(10^5) + (2)(10^3) + (7)(10^2) + (4)(10^0)$?

$$\begin{aligned}&(3)(10^6) + (9)(10^5) + (2)(10^3) + (7)(10^2) + (4)(10^0)\\ =&3000000 + 900000 + 2000 + 700 + 4\\ =&3902704\end{aligned}$$

**$600** What is 3119 in binary (base 2)?

| Division | Remainder |
|---|---|
| $3119 \div 2 = 1559$ | 1 |
| $1559 \div 2 = 779$ | 1 |
| $779 \div 2 = 389$ | 1 |
| $389 \div 2 = 194$ | 1 |
| $194 \div 2 = 97$ | 0 |
| $97 \div 2 = 48$ | 1 |
| $48 \div 2 = 24$ | 0 |
| $24 \div 2 = 12$ | 0 |
| $12 \div 2 = 6$ | 0 |
| $6 \div 2 = 3$ | 0 |
| $13 \div 2 = 1$ | 1 |
| $1 \div 2 = 0$ | 1 |

Thus, $3119_{10} = 110000101111_2$

**$800** What is this hexadecimal (base 16) number in decimal (base 10): $B8FE$?

$$\begin{aligned} B8FE &\to (11)(16^3) + (8)(16^2) + (15)(16^1) + (14)(16^0) \\ &= 45056 + 2048 + 240 + 14 \\ &= 47358 \end{aligned}$$

**$1000** Convert this hexadecimal (base 16) number to base 11: $D3C$?

$$\begin{aligned} D3C &\to (13)(16^2) + (3)(16^2) + (12)(16^10) \\ &= 3328 + 48 + 12 \\ &= 3388_{10} \end{aligned}$$

| Division | Remainder |
|---|---|
| $3388 \div 11 = 308$ | 0 |
| $308 \div 11 = 28$ | 0 |
| $28 \div 11 = 2$ | 6 |
| $2 \div 11 = 0$ | 2 |

Thus, $D3C_{16} = 3388_{10} = 2600_{11}$

## Random

**$200** What is $(1+2)^2 - (1^2 + 2^2)$?

(a) 14 (b) 4 (c) 2 (d) 12

$$\begin{aligned} (1+2)^2 - (1^2 + 2^2) &= 3^2 - (1^2 + 2^2) \\ &= 9 - (1^2 + 2^2) \\ &= 9 - (1 + 4) \\ &= 9 - 5 \\ &= 4 \quad \implies \text{(b) } 4 \end{aligned}$$

**\$400** If $x = -4$ and $y = 4$, then which is the largest?

(a) $\frac{x}{y}$ (b) $y - 1$ (c) $-xy$ (d) $x + y$

(a) $\frac{x}{y} = -\frac{4}{4} = -1$
(b) $y - 1 = 4 - 1 = 3$
(c) $-xy = -(-4)(4) = -(-16) = 16$
(d) $x + y = -4 + 4 = 0$
Thus, (c) $-xy$ is the largest.

**\$600** If $2^a = 8$ and $a = 3c$, then $c$ is:

(a) 6 (b) $\frac{3}{4}$ (c) 1 (d) $\frac{4}{3}$

Since we know that $\sqrt{8} = 3$, then $2^3 = 8$ so $a = 3$.

We also know that $3 = 3c$, therefore $c = 1$.

Thus, the answer ic (c) 1.

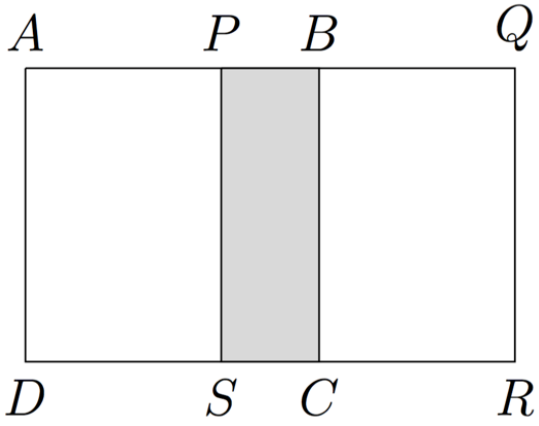
**\$800** 2 identical squares, $ABCD$ and $PQRS$, have side lengths 12. They overlap to form the 12 by 20 rectangle $AQRD$ shown below. What is the area of the shaded rectangle $PBCS$?

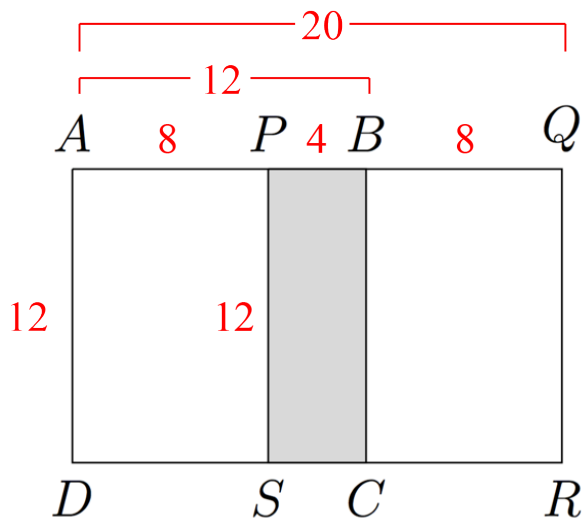
(a) 24

(b) 72

(c) 96

(d) 48






$$AD = 12 \qquad AQ = 20 \qquad AB = 12$$

$$\begin{aligned} \text{Thus, } BQ &= AQ - AB \\ &= 20 - 12 \\ &= 8 \end{aligned}$$

$$\begin{aligned} \text{So, } AP &= BQ = 8 \\ \text{Thus, } PB &= AQ - AP - BQ \\ &= 20 - 8 - 8 \\ &= 4 \end{aligned}$$

$$PS = 12$$

$$\text{Thus, Area} = 4 \times 12 = 48$$

Thus, the answer is (d) 48

**$1000** Andrea has finished the 3rd day of a 6 day canoe trip. If she has completed $\frac{3}{7}$ of the trip's total distance of 168 km. How many km per day must she average for the remainder of the trip?

(a) 29 km (b) 32 km (c) 24 km (d) 26 km

So far, Andrea has canoed $\frac{3}{7} \times 168 = 72$ km.

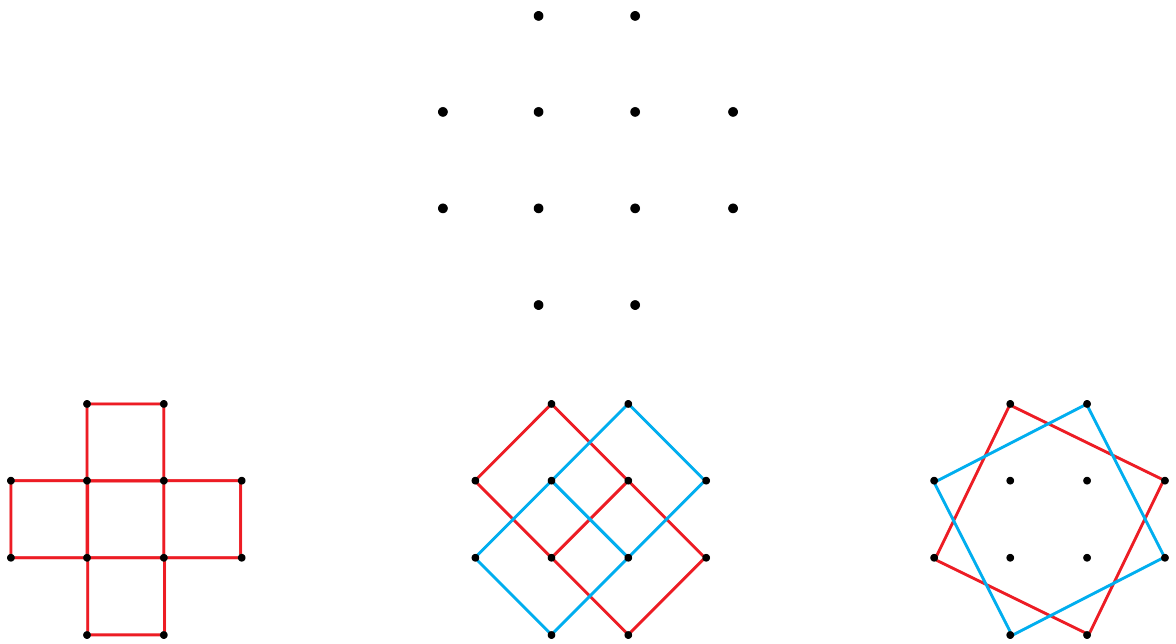
Thus, she must canoe $168 - 72 = 96$ km further in $6 - 3 = 3$ days.

Therefore, she must average $96 \div 3 = 32$ km per day for the remainder of her trip.

Thus, the answer is (b) 32 km per day.

## Final Jeopardy

12 points are marked on a rectangular grid. How many squares can be made by connecting these points?



There are $5 + 4 + 2 = 11$ squares total.