UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

April 4 & 5, 2017

Gauss Contest Preparation

General Information

The Gauss contest is an opportunity for students from grades 7 to 8 to have fun and challenge their mathematical problem solving skills. It is one of the many math contests that the University of Waterloo offers for students worldwide!

Registration Deadline: April 21, 2017

Contest Date: May 10, 2017

How can I prepare for the contest?

Practice! Work on a bunch of problems to improve your problem-solving skills. There are many resources available for you online. The best resource for practicing for the contest is by working on past contests! You have free access to all previous contests and solutions on the CEMC website: `http://www.cemc.uwaterloo.ca/contests/past_contests.html`

Problem Solving Strategies for Success

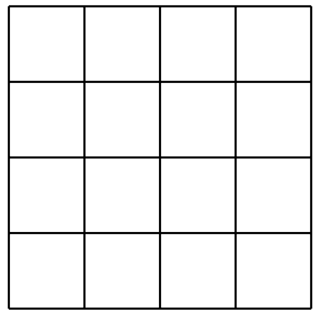
Make a list of problem-solving strategies!

- **ELIMINATE** the choices you know are not correct
- **DRAW** diagrams to visually represent the problem
- **PRACTICE** by working on problems from past contests
- **LEARN** different techniques and shortcuts from past contest solutions
- **MOVE ON** if you get stuck on a question. Use your time more effectively by working on other problems
- **READ** the problem, read it again, read it thrice! Make sure you know what the problem is asking for you to solve

Warm Up Questions

1. **How Many Squares?**

   In the grid below, how many squares are there in total?



Solution 1

Count the squares! Since we have a 4 by 4 grid, we can have 1 by 1, 2 by 2, 3 by 3, or 4 by 4 squares.

$$\begin{array}{lr} \text{1 by 1s:} & 16 \\ \text{2 by 2s:} & 9 \\ \text{3 by 3s:} & 4 \\ \text{4 by 4s:} & \underline{+\ 1} \\ & 30 \end{array}$$

And so there are 30 squares in total.

Solution 2

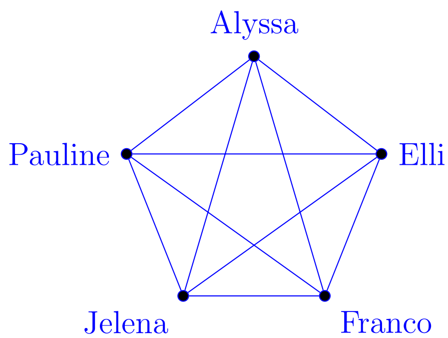
Notice there is 1 (4 by 4) square, 4 (3 by 3) squares, and 9 (2 by 2) squares. 1, 4, and 9 are all perfect squares! Following this pattern, there must be $4^2 = 16$ 1 by 1 squares. So there are $1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$ squares.

2. **Let's Be Friends!**

Alyssa, Elli, Franco, Jelena, and Pauline recently joined Facebook and decide to all friend each other. How many friendships are there?

Solution 1

We can draw the friendships and count them as shown below.



In the regular pentagon show above, there are 5 edges and 5 diagonals. And so, there are $5 + 5 = 10$ friendships.

Solution 2

Suppose we start by counting the number of friends Alyssa can add on Facebook. She can make friends with 4 people (Elli, Franco, Jelena, and Pauline). Next, Elli can make friends with 3 people (Franco, Jelena, and Pauline). Franco can make friends with 2 people (Jelena and Pauline). And finally, Jelena can be friends with 1 person (Pauline). Therefore, there are $4 + 3 + 2 + 1 = 10$ friendships.

Solution 3

There are 5 people and each person can become friends with 4 other people, so there are $4 \times 5 = 20$ possible friendships. However, this count is double counting the number of unique friendships there are. For example, Alyssa becoming friends with Elli and Elli becoming friends with Alyssa is the same friendship however it is counted as two friendships in the count. To remove the double counted friendships, the count is divided by 2. Thus, there are $\frac{4 \times 5}{2} = \frac{20}{2} = 10$ friendships.

Gauss Contest Problems

1. (2016 Grade 7 Gauss # 4)
   Which of these fractions is larger than $\frac{1}{2}$?

   (a) $\frac{2}{5}$ (b) $\frac{3}{7}$ (c) $\frac{4}{7}$ (d) $\frac{3}{8}$ (e) $\frac{4}{9}$

   Solution 1

   Use trial and error! Compare each answer to $\frac{1}{2} = 0.5$. Converting each answers gives...

   $\frac{2}{5} = 0.2$ $\frac{3}{7} = 0.4285$ $\frac{4}{7} = 0.5714$ $\frac{3}{8} = 0.375$ $\frac{4}{9} = 0.4444$

   $\frac{4}{7}$ is the only answer larger than $\frac{1}{2}$.

   Solution 2

   Note that $\frac{1}{2}$ is a half so we can check if the numerator in an answer is more than half of the denominator. $2 < 2.5, 3 < 3.5, 3 < 4$, and $4 < 4.5$ so (a), (b), (d), and (e) are eliminated respectively. This leaves us with (c) as the only possible solution. $4 > 3.5$.

   ANSWER: (c)

2. (2014 Grade 8 Gauss # 1)
   The number 10 101 is equal to

   (a) 1000 + 100 + 1 (b) 1000 + 10 + 1 (c) 10 000 + 10 + 1
   (d) 10 000 + 100 + 1 (e) 100 000 + 100 + 1

   Solution 1

   Evaluate each answer to find which one equals 10 101.

   Solution 2

   Using place values, 10 101 is the sum of one 10 000, one 100, and one 1.

   ANSWER: (d)

3. (2014 Grade 8 Gauss # 7)
How many prime numbers are there between 10 and 30?

(a) 4 (b) 7 (c) 6 (d) 3 (e) 5

Solution

A prime number is divisible by exactly two numbers, 1 and itself.

$\{12, 14, 16, 18, 20, 22, 24, 26, 28\}$ are *not* prime since they are even numbers and are divisible by at least 3 numbers, 1, 2, and itself.

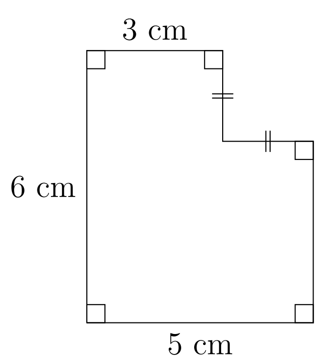
$\{15, 21, 27\}$ are also *not* prime since they are divisible by at least 1, 3, and itself.

25 is *not* prime since it is divisible by 1, 5, and itself.

The remaining numbers $\{11, 13, 17, 19, 23, 29\}$ are all prime. There are 6 prime numbers between 10 and 30.

ANSWER: (c)

4. (2004 Grade 7 Gauss # 10)
The perimeter of the figure, in cm, is



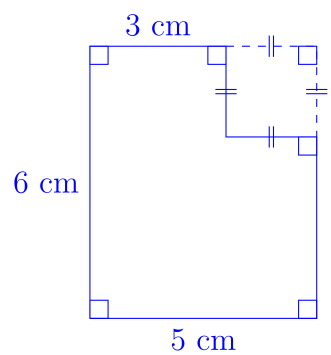
(a) 30 (b) 28 (c) 25 (d) 24 (e) 22

Solution 1

Note that the length of the smaller missing sides are equal. The length of the small missing sides is $5 - 3 = 2$ cm. Now, the length of the missing side on the right is $6 - 2 = 4$ cm. Therefore, the perimeter is $3 + 6 + 5 + 4 + 2 + 2 = 22$ cm.

Solution 2

If we complete the rectangle as shown below, we can visually see that the perimeter does not change whether or not the rectangle is completed so we can find the perimeter of the figure by finding the perimeter of the completed rectangle. Thus, the perimeter is $(2 \times 6) + (2 \times 5) = 12 + 10 = 22$ cm.



ANSWER: (e)

5. (2013 Grade 8 Gauss # 12)
The value of $(2^3)^2 - 4^3$ is

(a) 0 (b) -8 (c) 4 (d) 10 (e) 12

Solution 1

Using order of operations to evaluate, $(2^3)^2 - 4^3 = 8^2 - 4^3 = 64 - 64 = 0$

Solution 2

We first express 4 as 2^2 and then use exponent rules to evaluate.

$$\begin{aligned} (2^3)^2 - 4^3 &= (2^3)^2 - (2^2)^3 \\ &= 2^{3\times 2} - 2^{3\times 2} \\ &= 2^6 - 2^6 \\ &= 0 \end{aligned}$$

ANSWER: (a)

6. (2008 Grade 8 Gauss # 19)

In the addition of three-digit numbers shown below, the letters x and y represent different digits.

$$\begin{array}{cccc} & 3 & x & y \\ + & y & x & 3 \\ \hline 1 & x & 1 & x \end{array}$$

The value of $y - x$ is

(a) 3 (b) -5 (c) 7 (d) -7 (e) 2

Solution 1

After some trial and error, you might discover that $x = 0$ and $y = 7$ works, since $307 + 703 = 1010$. Therefore, since we are asked for the unique value of $y - x$, it must be $7 - 0 = 7$

Solution 2

When performing this addition, in the units column either $y + 3 = x$ or $y + 3 = x$ with a carry of 1, meaning that $y + 3 = 10 + x$. Therefore, either $y - x = -3$ or $y - x = 10 - 3 = 7$.

If there was no carry, then adding up the tens digits, we would get $x + x$ ending in a 1, which is impossible as $x + x = 2x$ which is even. Therefore the addition $y + 3$ must have a carry of 1. Therefore, $y - x = 7$.

ANSWER: (c)

7. (2004 Grade 7 Gauss # 13)

Two positive integers have a sum of 11. The greatest possible product of these two positive integers is

(a) 11 (b) 18 (c) 28 (d) 35 (e) 30

Solution 1

Make a table listing the two integers, a and b, that add to 11 and the product.

a	b	$a \times b$
1	10	10
2	9	18
3	8	24
4	7	28
5	6	30

The two positive integers must be 5 and 6 which gives the largest product of 30.

Solution 2

We need two positive integers, a and b, that add to 11 and have the largest product. The smaller the difference between a and b is, the larger $a \times b$ is. In this case, 5 and 6 add to 11 and have the smallest difference of 1. So, 5 and 6 must be the two positive integers. The product is 30.

ANSWER: (e)

8. (2008 Grade 8 Gauss # 15)

Abby has 23 coins. The coins have a total value of \$4.55. If she only has quarters (worth 25 cents each) and nickels (worth 5 cents each), how many quarters does she have?

(a) 15 (b) 17 (c) 18 (d) 16 (e) 21

Solution

If Abby had 23 nickels, the total value would be $23 \times \$0.05 = \$1.15$. But the total value of Abby's coins is \$4.55, which is \$3.40 more. Since a quarter is worth 20 cents more than a nickel, then every time a nickel is replaced by a quarter, the total value of the coins increase by 20 cents. For the total value to increase by \$3.40, we must replace $\$3.40 \div \$0.20 = 17$ nickels with quarters. Therefore, Abby has 17 quarters. (To check, if Abby has 17 quarters and 6 nickels, the total value of the coins that she ahs is $17 \times \$0.25 + 6 \times \$0.05 = \$4.25 + \$0.30 = \$4.55$.

ANSWER: (b)

9. (2014 Grade 8 Gauss # 14)

Betty is making a sundae. She must randomly choose one flavour of ice cream (chocolate or vanilla or strawberry), one syrup (butterscotch or fudge) and one topping (cherry or banana or pineapple). What is the probability that she will choose a sundae with vanilla ice cream, fudge syrup and banana topping?

(a) $\frac{1}{18}$ (b) $\frac{1}{6}$ (c) $\frac{1}{8}$ (d) $\frac{1}{9}$ (e) $\frac{1}{12}$

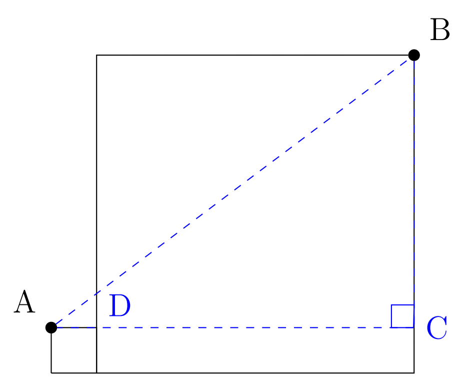
Solution

Betty has 3 equally likely choices for the flavour of ice cream (chocolate or vanilla or strawberry). For each of these 3 choices, she has 2 equally likely choices for the syrup (butterscotch or fudge). Thus, Betty has $3 \times 2 = 6$ equally likely choices for the flavour and syrup of the sundae. For each of these 6 choices, Betty has 3 equally likely choices for the topping (cherry or banana or pineapple). This makes $6 \times 3 = 18$ equally likely choices for her sundae. Since only 1 of these 18 choices is the sundae with vanilla ice cream, fudge syrup and banana topping, then the probability that Betty randomly chooses this sundae is $\frac{1}{18}$.

ANSWER: (a)

10. (2001 Grade 8 Gauss # 18)

Two squares are positioned, as shown. The smaller square has side length 1 and the larger square has side length 7. The length of AB is



(a) 14 (b) $\sqrt{113}$ (c) 10 (d) $\sqrt{85}$ (e) $\sqrt{72}$

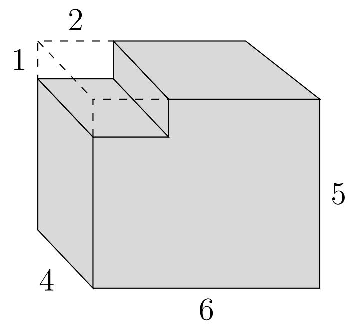
Solution

From A, along the side of the small square, we extend side AD so that it meets the big square at C as shown. The length of AC is the sum of the side lengths of the squares, i.e. $7 + 1 = 8$. The length of BC is the difference of the side lengths so $BC = 6$. By Pythagorean Theorem, $AB^2 = 8^2 + 6^2 = 100$, so $AB = 10$.

ANSWER: (c)

11. (2004 Grade 8 Gauss # 15)

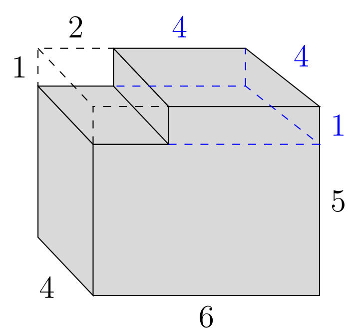
In the diagram, the volume of the shaded solid is



(a) 8 (b) 112 (c) 113 (d) 120 (e) 128

Solution 1

Break the shaded solid into two rectangular prisms as shown below. The volume of the shaded solid can be found by adding the volumes of these two prisms.



The larger rectangular prism is 4 by 6 by 4 so the volume is $4 \times 6 \times 4 = 96$. The smaller rectangular prism is 4 by 4 by 1 so the volume is $4 \times 4 \times 1 = 16$. Thus, the volume of the shaded solid is $96 + 16 = 112$.

Solution 2

To find the volume of the shaded solid, we can subtract the volume of the missing corner from the volume of the whole rectangular prism. The whole prism is 4 by 6 by 5 so the volume is $4 \times 6 \times 5 = 120$. Next, the missing corner is 4 by 2 by 1 so the volume is $4 \times 2 \times 1 = 8$. Thus, the volume is $120 - 8 = 112$.

ANSWER: (b)

12. (2011 Grade 8 Gauss # 19)
How many positive integers less than 400 can be created using only the digits 1, 2, or 3, with repetition of digits allowed?

(a) 30 (b) 33 (c) 36 (d) 39 (e) 42

Solution

We need to consider the three following cases:

Case # 1: 1-digit integers
Since 1, 2, and 3 are the only digits that may be used, there are only 3 possible 1-digit integers less than 400 (namely 1, 2, and 3).

Case # 2: 2-digit integers
There are three possible choices for the first digit and three possible choices for the second digit, so there are $3 \times 3 = 9$ possible 2-digit integers less than 400 (i.e. 11, 12, 13, 21, 22, 23, 31, 32, 33).

Case # 3: 3-digit integers
There are three choices for the first digit, three choices for the second digit, and three choices for the third digit, so there are $3 \times 3 \times 3 = 27$ possible 3-digit integers less than 400 (i.e. 111, 112, 113, 121, 122, 123, 131, 132, 133, 211, 212, 213, 221, 222, 223, 231, 232, 233, 311, 312, 313, 321, 322, 323, 331, 332, 333).

In total, the number of positive integers less than 400 that can be created using only digits 1, 2, and 3 (with repetition allowed), is $3 + 9 + 27 = 39$.

ANSWER: (d)

13. (2016 Grade 8 Gauss # 15)

A map has a scale of 1:600 000. On the map, the distance between Gausstown and Piville is 2 cm. What is the actual distance between the towns?

(a) 12 km (b) 1.2 km (c) 120 km (d) 1200 km (e) 12 000 km

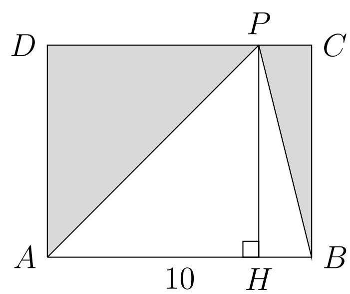
Solution

The map's scale of 1:600 000 means that a 1 cm distance on the map represents an actual distance of 600 000 cm. So then 2 cm measured on the map represents an actual distance of $2\times$ 600 000 $=$ 1 200 000 cm or 12 000 m or 12 km. The actual distance between Gausstown and Piville is 12 km.

ANSWER: (a)

14. (2014 Grade 8 Gauss # 16)

In the diagram, $ABCD$ is a rectangle.



If the area of the triangle ABP is 40, then the area of the shaded region is

(a) 20 (b) 40 (c) 60 (d) 50 (e) 80

Solution 1

Construct PH perpendicular to AB, as shown. The area of $\triangle ABP$ is 40 and so $\frac{1}{2} \times AB \times PH = 40$, or $AB \times PH = 80$, and since $AB = 10$, then $PH = 8$. Since $CB = PH = 8$, the area of $ABCD$ is $10 \times 8 = 80$. The shaded area equals the area of $\triangle ABP$ subtracted from the area of $ABCD$, or $80 - 40 = 40$.

Solution 2

As in Solution 1, we construct PH perpendicular to AB. Since both DA and CB are perpendicular to AB, then PH is parallel to DA and to CB. That is, $DAHP$ and $PHBC$ are rectangles. Diagonal PA divides the area of rectangle $DAHP$ into two equal areas, $\triangle PAH$ and $\triangle PAD$. Diagonal PB divides the area of rectangle $PHBC$ into two equal areas, $\triangle PBH$ and $\triangle PBC$. Therefore, the area of $\triangle PAH$ added to the area of $\triangle PBH$ is equal to the area of $\triangle PAD$ added to the area of $\triangle PBC$. However, the area of $\triangle PAH$ added to the area of $\triangle PBH$ is equal to the area of $\triangle ABP$, which is 40. So the area of $\triangle PAD$ added to the area of $\triangle PBC$ is also 40. Therefore, the area of the shaded region is 40.

ANSWER: (b)

15. (2005 Grade 8 Gauss # 17)
If a is an even integer and b is an odd integer, which of the following could represent an odd integer?

(a) ab (b) $a+2b$ (c) $2a-2b$ (d) $a+b+1$ (e) $a-b$

Solution 1

In this solution, we try specific values. This does not guarantee correctness, but it will tell us which answers are wrong. We try setting $a=2$ (which is even) and $b=1$ (which is odd). Then,

$$\begin{aligned} ab &= 2 \times 1 = 2 \\ a+2b &= 2+2(1) = 4 \\ 2a-2b &= 2(2)-2(1) = 2 \\ a+b+1 &= 2+1+1 = 4 \\ a-b &= 2-1 = 1 \end{aligned}$$

Therefore, $a-b$ is the only choice which gives an odd answer.

Solution 2

Since a is even, then ab is even, since an even integer times any integer is even.

Since a is even and $2b$ is even (since 2 times any integer is even), then their sum $a+2b$ is even.

Since 2 times any integer is even, then both $2a$ and $2b$ are both even, so their difference $2a-2b$ is even (since even minus even is even).

Since a is even and b is odd, then $a+b$ is odd, so $a+b+1$ is even.

Since a is even and b is odd, then $a-b$ is odd.

Therefore, $a-b$ is the only choice which gives an odd answer.

ANSWER: (e)

16. (2007 Grade 8 Gauss # 20)

Lorri took a 240 km trip to Waterloo. On her way there, her average speed was 120 km/h. She was stopped for speeding, so on her way home, her average speed was 80 km/h. What was her average speed, in km/h, for the entire round-trip?

(a) 90 (b) 96 (c) 108 (d) 102 (e) 110

Solution 1

Lorri's 240 km trip to Waterloo at 120 km/h took $240 \div 120 = 2$ hours. Lorri's 240 km trip home took $240 \div 80 = 3$ hours. In total, Lorri drove 480 km in 5 hours, for an average speed of $\frac{480}{5} = 96$ km/h.

Solution 2

Lorri's 240 km trip to Waterloo at 120 km/h took $240 \div 120 = 2$ hours. Lorri's 240 km trip home took $240 \div 80 = 3$ hours. Over the 5 hours that Lorri drove, her speeds were 120, 120, 80, 80, and 80, so her average speed was

$$\frac{120+120+80+80+80}{5} = \frac{480}{5} = 96 \text{ km/h}.$$

ANSWER: (b)

17. (2007 Grade 8 Gauss # 15)

Sally picks four consecutive positive integers. She divides each integer by four, and then adds the remainders together. The sum of the remainders is

(a) 6 (b) 1 (c) 2 (d) 3 (e) 4

Solution

Since Sally picked four consecutive positive integers, two integers must be even and the other two must be odd. Of the even integers, one will have remainder 0 and the other integer will have remainder 2 when divided by four. Of the odd integers, one will have remainder 1 and the other will have remainder 3. Thus, the sum of the remainders are $0 + 2 + 1 + 3 = 6$.

ANSWER: (a)

18. (2009 Grade 7 Gauss # 21)

Lara ate $\frac{1}{4}$ of a pie and Ryan ate $\frac{3}{10}$ of the same pie. The next day Cassie ate $\frac{2}{3}$ of the pie that was left. What fraction of the original pie was not eaten?

(a) $\dfrac{9}{10}$ (b) $\dfrac{3}{10}$ (c) $\dfrac{7}{60}$ (d) $\dfrac{3}{20}$ (e) $\dfrac{1}{20}$

Solution

Together, Lara and Ryan ate $\frac{1}{4} + \frac{3}{10} = \frac{5}{20} + \frac{6}{10} = \frac{11}{20}$ of the pie. Therefore, $1 - \frac{11}{20} = \frac{9}{20}$ of the pie remained. The next day, Cassie ate $\frac{2}{3}$ of the pie that remained. This implie that $1-\frac{2}{3} = \frac{1}{3}$ of the pie that was remaining was left after Cassie finished eating. Thus, $\frac{1}{3}$ of $\frac{9}{20}$, or $\frac{3}{20}$ of the original pie was not eaten.

ANSWER: (d)

19. (2004 Grade 7 Gauss # 22)

The entire contents of a jug can exactly fill 9 small glasses and 4 large glasses of juice. The entire contents of the jug could instead fill 6 small glasses and 6 large glasses. If the entire contents of the jug is used to fill only large glasses, the maximum number of large glasses that can be filled is

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

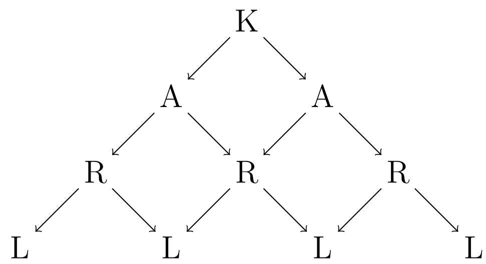
Solution

Let the volume of a large glass be L and of a small glass be S. Since the jug can exactly fill either 9 small glasses and 4 large glasses, or 6 small glasses and 6 large glasses, then $9S + 4L = 6S + 6L$ or $3S = 2L$. In other words, the volume of 3 small glasses equals the volume of 2 large glasses. (We can also see this without using algebra. If we compare the two cases, we can see that if we remove 3 small glasses then we increase the volume by 2 large glasses.) Therefore, the volume of 9 small glasses equals the volume of 6 large glasses. Thus, the volume of 9 small glasses and 4 large glasses equals the volume of 6 large glasses and 4 large glasses, or 10 large glasses in total, and so the jug can fill 10 large glasses in total.

ANSWER: (c)

20. (2007 Grade 7 Gauss # 21)

In the diagram, how many paths can be taken to spell "KARL"?



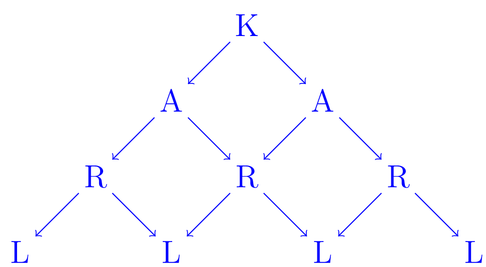
(a) 4 (b) 16 (c) 6 (d) 8 (e) 14

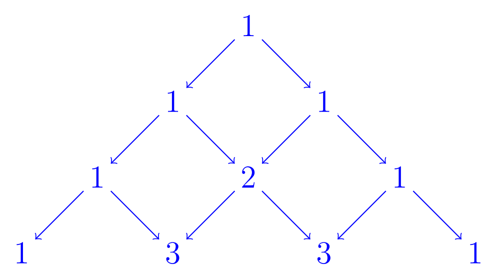
Solution 1

Starting at the "K", there are two possible paths that can be taken. At each "A" there are again two possible paths that can be taken. Similarly, at each "R" there are two possible paths that can be taken. Therefore, the total number of paths is $2 \times 2 \times 2 = 8$. (This can be checked by actually tracing out the paths.)

Solution 2

We can actually count the number of possible paths by using Pascal's Triangle.



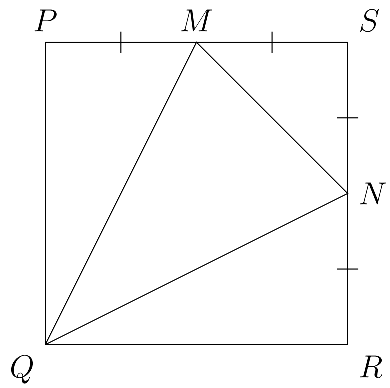


There is only 1 path to reach the first "L", 3 possible paths to reach the second "L", 3 possible paths to reach the third "L", and again only 1 path to reach the fourth "L". Therefore, the total number of possible paths to spell "KARL" is $1 + 3 + 3 + 1 = 8$.

ANSWER: (d)

21. (2015 Grade 8 Gauss # 21)

In square $PQRS$, M is the midpoint of PS and N is the midpoint of SR.



If the area of $\triangle SMN$ is 18, then the area of $\triangle QMN$ is

(a) 36 (b) 72 (c) 90 (d) 48 (e) 54

Solution

In square $PQRS$, $PS = SR$ and since M and N are midpoints of these sides having equal length, then $MS = SN$.

The area of $\triangle SMN$ is $\frac{1}{2} \times MS \times SN$.

Since this area equals 18, then $\frac{1}{2} \times MS \times SN = 18$ or $MS \times SN = 36$ and so $MS = SN = 6$ (since they are equal in length).

The side of the square, PS, is equal in length to $PM + MS = 6 + 6 = 12$ (since M is the midpoint of PS) and so $PS = SR = RQ = QP = 12$.

The area of $\triangle QMN$ is equal to the area of square $PQRS$ minus the combined areas of the three right-angled triangles, $\triangle SMN, \triangle NRQ$ and $\triangle QPM$.

Square $PQRS$ has area $PS \times SR = 12 \times 12 = 144$.

$\triangle SMN$ has area 18, as was given in the question.

$\triangle NRQ$ has area $\frac{1}{2} \times QR \times RN = \frac{1}{2} \times 12 \times 6 = 36$ (since $SN = RN = 6$).

$\triangle QPM$ has area $\frac{1}{2} \times QP \times PM = \frac{1}{2} \times 12 \times 6 = 36$.

Thus the area of $\triangle QMN$ is $144 - 18 - 36 - 36 = 54$.

ANSWER: (e)

22. (2007 Grade 8 Gauss # 23)

There are various ways to make \$207 using only \$2 coins and \$5 bills. One such way is using one \$2 coin and forty-one \$5 bills. Including this way, in how many different ways can \$207 be made using only \$2 coins and \$5 bills?

(a) 9 (b) 10 (c) 19 (d) 41 (e) 21

Solution 1

Using \$5 bills, any amount of money that is a multiple of 5 (that is, ending in a 5 or a 0) can be made. In order to get to \$207 from a multiple of 5 using only \$2 coins, the multiple of \$5 must end in a 5. (If it ended in a 0, adding \$2 coins would still give an amount of money that was an even integer, and so couldn't be \$207.)

Also, from any amount of money ending in a 5 that is less than \$207, enough \$2 coins can always be added to get to \$207. The positive multiples of 5 ending in a 5 that are less than 207 are 5, 15, 25, . . ., 195, 205. An easy way to count the numbers in this list is to remove the units digits (that is, the 5s) leaving 0, 1, 2, . . ., 19, 20; there are 21 numbers in this list.

These are the only 21 multiples of 5 from which we can use \$2 coins to get to \$207. So there are 21 different ways to make \$207.

Solution 2

We are told that 1 \$2 coin and 41 \$5 bills make \$207. We cannot use fewer \$2 coins, since 0 \$2 coins would not work, so we can only use more \$2 coins. To do this, we need to make change that is, trade \$5 bills for \$2 coins. We cannot trade 1 \$5 bill for \$2 coins, since 5 is not even. But we can trade 2 \$5 bills for 5 \$2 coins, since each is worth \$10.

Making this trade once gets 6 \$2 coins and 39 \$5 bills.

Making this trade again get 11 \$2 coins and 37 \$5 bills.

We can continue to do this trade until we have only 1 \$5 bill remaining (and so \$202 in \$2 coins, or 101 coins).

So the possible numbers of \$5 bills are 41, 39, 37, . . ., 3, 1. These are all of the odd numbers from 1 to 41. We can quickly count these to get 21 possible numbers of \$5 bills and so 21 possible ways to make \$207.

ANSWER: (e)

23. (2012 Grade 8 Gauss # 22)

In downtown Gaussville, there are three buildings with different heights: The Euclid (E), The Newton (N), and The Galileo (G). Only one of the statements below is true.

1. The Newton is not the shortest.
2. The Euclid is the tallest.
3. The Galileo is not the tallest.

Ordered from *shortest to tallest* in height, the buildings are

(a) N,G,E (b) G,E,N (c) E,N,G (d) N,E,G (e) E,G,N

Solution

Since only one of the given statements is true, then the other two statements are both false.

Assume that the second statement is true (then the other two are false). If the second statement is true, then The Euclid is the tallest building. Since the third statement is false, then The Galileo is the tallest building. However, The Euclid and The galileo cannot both be the tallest building so we have a contradiction. Therefore the second statement cannot be true.

Assume the third statement is true. Then The Galileo is not the tallest building. Since the second statement must be false, then The Euclid is not the tallest building. If both The Galileo and The Euclid are both not the tallest building, then The Newton must be the tallest. However, since the first statement must also be false, then The Newton is the shortest building and we have again reached a contradiction. Therefore, the third statement cannot be true.

Since the first statement is true, then The Newton is either the second tallest building or the tallest building. Since the third statement is false, then The Galileo is the talelst building which means that The Newton is the second tallest. Since the second statment is false, then The Euclid is not the tallest and therefore must be the shortest (since The Newton is the second tallest).

Ordered from shortest to tallest, the buildings are The Euclid (E), The Newton (N), and The Galileo (G).

ANSWER: (c)

24. (2002 Grade 8 Gauss # 24)

Veronica has 6 marks on her report card.

The mean of the 6 marks is 74.

The mode of the 6 marks is 76.

The median of the 6 marks is 76.

The lowest mark is 50.

The highest mark is 94.

Only one mark appears twice and no other mark appears more than twice.

Assuming all of her marks are integers, the number of possibilities for her second lowest mark is

(a) 17 (b) 16 (c) 25 (d) 18 (e) 24

Solution

Since the mode of Veronicas 6 marks is 76, and only one mark appears more than once (and no marks appear more than twice), then two of the marks must be 76. This tells us that four of her marks were 50, 76, 76, 94. Since the median of her marks is 76 and she has six marks in total (that is, an even number of marks), then the two marks of 76 must be 3rd and 4th when the marks are arranged in increasing order. Let the second lowest mark be M, and the second highest be N. So the second lowest mark M is between (but not equal to) 50 and 76, and the second highest mark N is between (but not equal to) 76 and 94. We still need to use the fact that the mean of Veronicas marks is 74, so

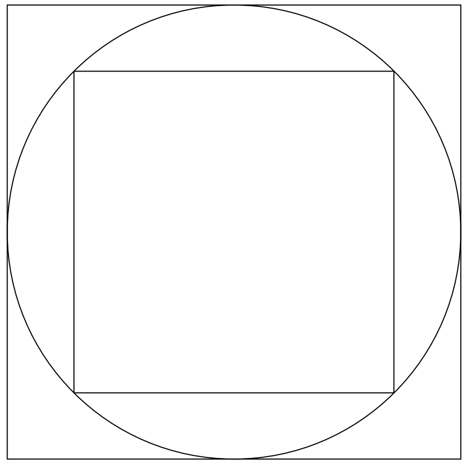
$$\begin{aligned}\frac{50 + M + 76 + 76 + N + 94}{6} &= 74\\ M + N + 296 &= 444\\ M + N &= 148\\ M &= 148 - N \ (*)\end{aligned}$$

We know already that M is one of 51 through 75, but the possibilities for N and the equation (*) restrict these possibilities further. Since N can be any of 77 through 93, there are exactly 17 possibilities for N. The largest value of M corresponds to $N = 77$ (ie. $M = 71$) and the smallest value for M is when $N = 93$ (ie. $M = 55$). Thus the possibilities for M are 55 through 71, ie. there are 17 possibilities in total for M, the second smallest mark.

ANSWER: (a)

25. (2005 Grade 8 Gauss # 21)

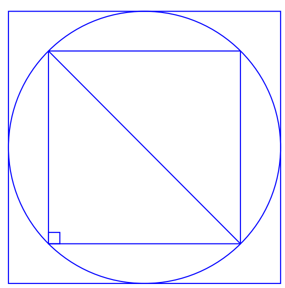
In the diagram, a circle is inscribed in a large square and a smaller square is inscribed in the circle.



If the area of the large square is 36, the area of the smaller square is

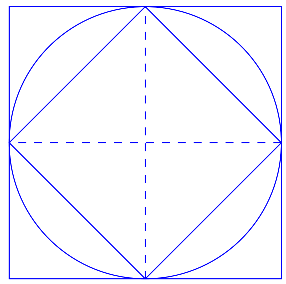
(a) 15 (b) 12 (c) 9 (d) 24 (e) 18

Solution 1



Since the area of the large square is 36, then the side length of the large square is 6 (because it is a 6×6 square). This means the diameter of the inscribed circle is 6 because the diameter is of equal length with the side of the large square. Notice that the diagonal of the smaller square is also equal to the diameter to the circle since both ends of the diagonal touch the circle. And so the diagonal of the smaller square is 6. By Pythagorean Theorem, the side length, s, of the smaller square is $s^2 + s^2 = 6^2 \Rightarrow 2s^2 = 36 \Rightarrow s^2 = 18 \Rightarrow s = \sqrt{18}$. And so the area of the smaller square is $s^2 = (\sqrt{18})^2 = 18$.

Solution 2



Rotate the smaller square so that its four corners are at the four points where the circle touches the large square. Next, join the top and bottom points where the large square and circle touch, and join the left and right points. By symmetry, these two lines divide the large square into four sections (each of which is square) of equal area. But the original smaller square occupies exactly one-half of each of these four sections, since each edge of the smaller square is a diagonal of one of these sections. Therefore, the area of the smaller square is exactly one-half of the area of the larger square, or 18.

ANSWER: (e)