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Grade 7/8 Math Circles

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Game Theory

Motivating Problem:

Roger and Colleen have been placed in two separate interrogation rooms as suspects for breaking the vending machine in DC. They are both guilty, but before committing the crime they agreed that they would both stay quiet if they were to get caught. However, all communication has been lost between the two of them and they are now given the following possible outcomes depending on whether they choose to confess to the crime or stay quiet:

- If they both confess, they will both serve 5 hours of detention.
- If one of them confesses and the other stays quiet, then the one who confessed will serve 0 hours of detention while the one who stayed quiet will serve 10 (the confession is that both of them broke the machine).
- If both of them stay quiet, they will both serve 2 hours of detention.

		Colleen	
		Confess	Stay Quiet
Roger	Confess	$(5, 5)$	$(0, 10)$
	Stay Quiet	$(10, 0)$	$(2, 2)$

What should Roger and Colleen do?

Making a Decision...

Roger and Colleen did have an agreement to both stay quiet, but now there is too much at risk for them to trust one another. They must consider what the other person might do to make sure they get the best possible outcome for themselves. Assuming now that Roger wants to minimize his detention time, he must consider the following:

- If Colleen confesses, then Roger's best decision is to **confess**.
- If Colleen stays quiet, then Roger's best decision is to **confess**.

Let's circle Rogers' decisions in red in the table below.

Looking at Colleen's point of view:

- If Roger is to confess, then Colleen's best decision is to **confess**.
- If Roger is to stay quiet, then Colleen's best decision is to **confess**.

Let's box Colleen's decisions in blue in the table below.

		Colleen	
		Confess	Stay Quiet
Roger	Confess	(5, 5)	(0, 10)
	Stay Quiet	(10, 0)	(2, 2)

You may think, "Why wouldn't they both stay quiet?", as it would lead to a better outcome than if they were to both confess. But, if Roger decides to keep their agreement and Colleen decides to break it, then Roger is going to serve 10 hours in detention. Therefore, by using this same logic for Colleen, choosing to **confess** is the best(safest) option for both of them.

Game Theory - The Prisoner's Dilemma

		Colleen	
		Confess	Stay Quiet
Roger	Confess	$(\boxed{5, 5})$	(0, 10)
	Stay Quiet	(10, 0)	(2, 2)

Game Theory is the study of deciding on the best strategy to win a game, and applies to games where the decisions of one person affect the decisions of the other(s). When we say "win a game," this can also be thought of as just satisfying our interests as much as possible, given a certain situation (game). In the "Prisoner's Dilemma" example, to the win the game meant to get the least amount of detention time as possible. In trying to do so, we found that Roger and Colleen should both confess. This outcome, where they both serve a detention sentence of 5 hours each, is called a *Nash Equilibrium*.

Nash Equilibrium

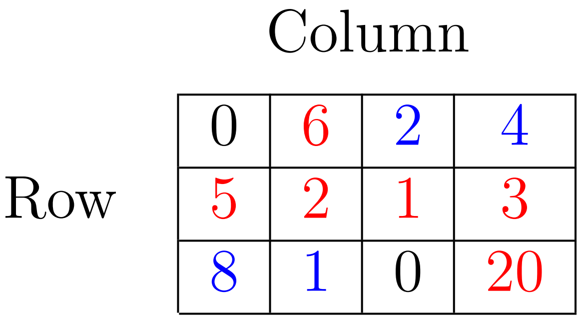
The Nash Equilibrium is the outcome in a game where one player cannot become better off if they individually change their decision. To find this outcome you take the following steps:

- Consider each of the column player's decisions, and based on each of those decisions, mark the outcome that corresponds to the row player's best decision.
- Consider each of the row player's decisions, and based on each of those decisions, mark the outcome that corresponds to the column player's best decision.
- For the outcomes in the game where the markings you have made intersect, you have a Nash Equilibrium!

Example. Find the Nash Equilibrium

Here are the rules for the game below:

- The row player will choose a row and the column player will choose a column.
- Combining these two choices, we land on one square in the table.
- Positive red values are payouts, in dollars, to the row player.
- Negative blue values are payouts, in dollars, to the column player (-2 is pays the column player $2).
- All payouts are made by the losing player (loser pays the winner).



Column

Row

0	6	2	4
5	2	1	3
8	1	0	20

The row player's best response given the column's player decisions:

- If the column player chooses column 1, then the row player should choose **row 2**.
- If the column player chooses column 2, then the row player should choose **row 1**.
- If the column player chooses column 3, then the row player should choose **row 2**.
- If the column player chooses column 4, then the row player should choose **row 3**.

The row player's best possible outcomes are circled in red below:

	Column			
	0	⑥	2	4
Row	⑤	2	①	3
	8	1	0	⑳

Repeating these steps from the point of view of the column player:

- If the row player chooses row 1, then the column player should choose **column 4**.
- If the row player chooses row 2, then the column player should choose **column 3**.
- If the row player chooses row 3, then the column player should choose **column 1**.

The column player's best possible outcomes are boxed in blue below:

	Column			
	0	⑥	2	$\boxed{4}$
Row	⑤	2	$\boxed{\text{①}}$	3
	$\boxed{8}$	1	0	⑳

Fair and Unfair Games

To check whether a game is fair or unfair we must find the *best strategy* of the game.

Best Strategy - the decision in a game that leads to the outcome that is a Nash Equilibrium.

A game is considered **fair** if the *best strategy* of the game leads to an outcome that is just as good for both players. If not, the game is considered **unfair**

Is the game in the previous example considered fair?

No, because the best strategy of both players will lead to an outcome of $1 to the row player.

Bonus Mark Game

There will be a Gauss Contest preparation test during week 8 of Math Circles this year. Luckily for you, Math Circles is offering up bonus marks! However, instead of just giving you bonus marks, you're going to earn these bonus marks by playing a game. The rules are as follows:

- On the count of three you can choose to raise your hand or not.
- If everyone raises their hand, the whole class gets two bonus marks.
- If at least one person does not raise their hand, then everyone that did receives 0 bonus marks.
- If you don't raise your hand, you are guaranteed one bonus mark.

		Rest of Class	
		Hand Up	Hand Down
You	Hand Up	$(2, 2)$	$(0, 1)$
	Hand Down	$(1, 0)$	$(1, 1)$

Let's find the Nash Equilibrium of this game:

- If the rest of the class puts their hand up, then your best decision is to **put your hand up**.
- If the rest of the class keeps their hand down, then your best decision is to **keep your hand down**.

From the rest of the classes point of view (assuming the rest of the class acts as one person, making one decision):

- If you put your hand up, then the best decision for the rest of the class is to **put their hand up**.
- If you keep your hand down, then the best decision for the rest of the class to **keep their hand down**.

		Rest of Class	
		Hand Up	Hand Down
You	Hand Up	$(\boxed{2, 2})$	$(0, 1)$
	Hand Down	$(1, 0)$	$(\boxed{1, 1})$

Notice that we have two Nash Equilibriums for this game! From each of these outcomes, neither player can individually change their decision to get a better outcome. In particular, if the rest of the class has their hand up, you would not want to keep your hand down, because then you would only receive 1 bonus mark instead of 2. Also, if the rest of the class keeps their hand down, you would not want to then put your hand up, because then you would receive 0 bonus marks instead of 1.

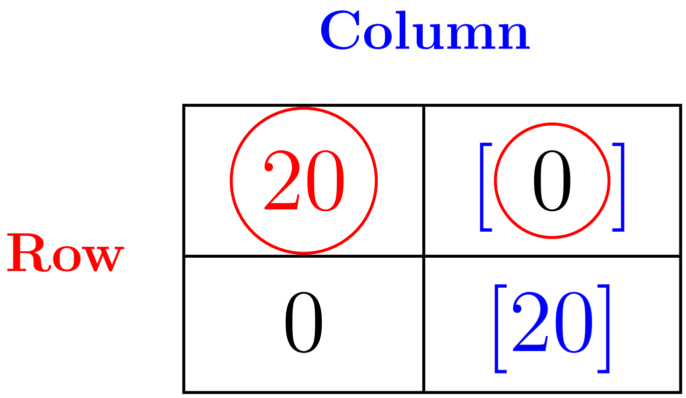
In the case that their are two Nash Equilibriums, we have think about what is good about making either decision and what is bad about making either decision, given the rules of the game. When we played this game in class, about half of the class put their hands up, while the other half kept their hands down. The students who put their hands up argued that if their is a chance to receive 2 bonus marks, which is more than 1, then everyone should simply just put their hands up. However, given the rules of the game, since all it took was for one student to keep their hand down and cause the students who put their hands up to receive 0 bonus marks, there was too much risk in trying to get the 2 bonus marks. Therefore, although there may be two Nash Equilibrium, or 2 “best” decisions for all players to make, you must consider the rules of the game while trying to make a decision.

What if the rule was, “If half of the class keeps their hand down, the students who put their hands up receive 0 bonus marks”. Does this change your thinking?

Problem Set

1. Look back to the game we played in the "**Example - Find the Nash Equilibrium**" section of the handout. What are the Nash Equilibriums of these games? Are these games fair?

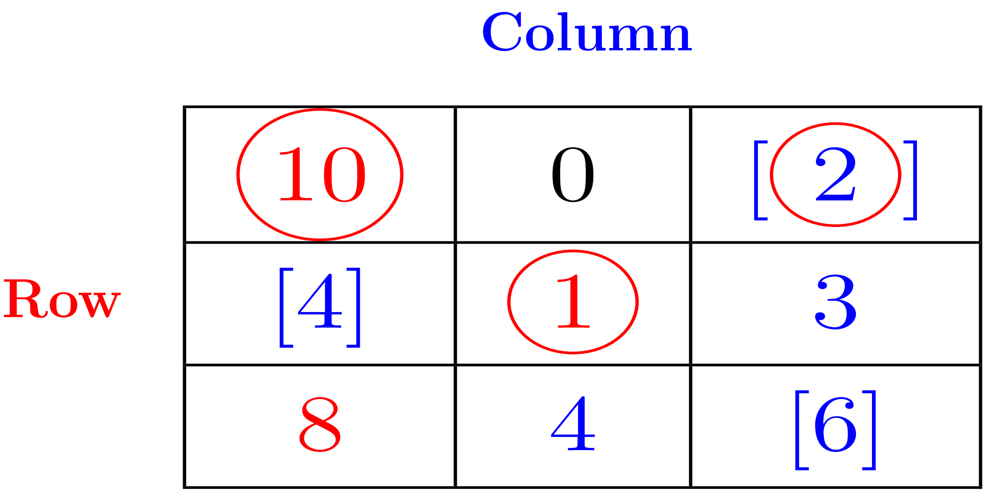
   (a)



	Column	
Row	20	[0]
	0	[20]

   The Nash Equilibrium of this game is a payout of $0. Therefore, this game is fair.

   (b)



		Column	
	10	0	[2]
Row	[4]	1	3
	8	4	[6]

   The Nash Equilibrium of this game is a payout of $2 to the column player. Therefore, this game is unfair.

(c)

Column

Row	3	[4]
	[2]	0
	[1]	2
	④	[③]

The Nash Equilibrium is a payout of $3 to the row player. Therefore, this game is unfair.

(d)

Column

Row	3	1	④	[7]
	1	[⓪]	2	3
	⑤	2	[3]	0
	3	[⓪]	1	⑤

The Nash Equilibriums are a payouts of $0. Therefore, this game is fair.

2. Complete the table below which represents the game, “Rock, Paper, Scissors”. Treat a win as having a value of 1 (in red) for the Row Player, a win as having a value of 1 (in blue) for the Column Player, and treat a draw as having a value of zero. If there is one, find a best pure strategy for this game. If not, explain what this could mean when it comes to deciding on a strategy.

Column

		Rock	Paper	Scissors
Row	Rock	0	[1]	①
	Paper	①	0	[1]
	Scissors	[1]	①	0

Answers may vary. Notice we could not find a *best strategy* for this game; therefore, considering the values for winning are the same for each outcome, the best strategy for both players is to decide what move they are going to make at random.

3. Alyssa and Tom are making treats for a bake sale to raise money for new calculators for the school. They are having a hard time deciding whether to make cookies or brownies, since the person who sells the most treats will win a prize. After doing some research within the school, they both find out the following information:

   - If Tom and Alyssa both make cookies, then Tom will sell 35 cookies and Alyssa will sell 45 cookies.
   - If Tom and Alyssa both make brownies, then Tom will sell 20 brownies and Alyssa will sell 30 brownies.
   - If Tom makes cookies and Alyssa makes brownies, then Tom will sell 60 cookies and Alyssa will sell 40 brownies.
   - If Tom makes brownies and Alyssa makes cookies, then Tom will sell 30 brownies and Alyssa will sell 70 cookies.

Alyssa

		Cookies	Brownies
Tom	Cookies	35,45	60,40
	Brownies	30,70	20,30

Using the information above to fill in the table with how many treats they each will sell depending on what treat they decide to make. Then

(a) Find the Nash Equilibrium of this game to find out what type of treat each student will make.

Alyssa

		Cookies	Brownies
Tom	Cookies	[35,45]	60,40
	Brownies	30,70	20,30

The Nash Equilibrium of this game tells that that both students will make cookies.

(b) Is this a fair competition?

Since the best strategy for both students does not lead to an equal outcome, this competition is not fair.

(c) If not, what can Tom do to ensure the bake sale is as successful as possible, regardless if he wins or not?

Let's make another game table to see the combined amount of treats both students will sell together.

Alyssa

		Cookies	Brownies
Tom	Cookies	80	100
	Brownies	100	50

Therefore, Tom should propose to Alyssa that one of them makes cookies and the other makes brownies.

4. The game is defined as follows:

   - Two hunters go out to catch meat.
   - There are two rabbits in the range and one stag. The hunters can each bring the equipment necessary to catch one type of animal.
   - The stag has more meat than the rabbits combined, but both hunters must chase the stag to catch it.
   - Rabbit hunters can catch all of their prey by themselves.
   - The values in the table represent the amount of meat (in pounds) the hunters will get depending on what they both choose to hunt.
   - Each rabbit provides 1lb of meat and each stag provides 6lbs of meat.

		Hunter 2	
		Stag	Rabbit
Hunter 1	Stag	3,3	0,2
	Rabbit	2,0	1,1

Find the Nash Equilibrium(s) of this game. If there is more than one, what are the pros and cons of choosing the corresponding strategies.

First we find the Nash Equilibrium for this game:

		Hunter 2	
		Stag	Rabbit
Hunter 1	Stag	[3,3]	0,2
	Rabbit	2,0	[1,1]

Notice that for both of the outcomes marked above, neither hunter can become better off by individually changing their decision; therefore there are two Nash Equilibriums (Equilibria).

Choosing to hunt the stag is the best decision for both hunters, considering they would maximize the amount of meat they get. However, this decision could be bad, because there is a chance a hunter could get no meat if the other one decides to hunt rabbits.

Choosing to hunt the rabbits is the best decision for both hunters, considering there is no risk of getting no meat. In particular, the worst outcome that can occur is getting 1 pound of meat, and the best outcome is getting 2 pounds of meat. However, this decision is bad, considering both hunters want to maximize the amount of meat they get.

5. * You are the owner of a clothing store and you must decide at what price to sell a hot new suit. You know that your competitor across the street is selling the same suit, so you must take into consideration the price at which they are selling to make sure you attract as many customers as possible. You have the following information:

   - If you and your competitor both sell the suit at $50, you will sell 55% of the total number of suits sold between the two of you.
   - If you sell the suit at $50 but your competitor sells the suit at $70, you will sell 70% of the total number of suits sold between the two of you.
   - If you sell the suit at $70 but your competitor sells the suit at $50, you will sell %40 of the total number of suits sold between the two of you.
   - If you and your competitor both sell the suit at $70, you will sell %55 of the total number of suit sold between the two of you.

   Represent the above information as a game to answer the following questions:

Competitor

You

	$50	$70
$50		
$70		

(a) If you and your competitor's main goal is to sell as many suits as possible, what is the best price to sell the suit at?

Competitor

You

	$50	$70
$50	55%, 45%	70%, 30%
$70	40%, 60%	55%, 45%

Above is the game for what price to sell the suit at. We see the percentage of the total number of suits sold that will be sold by you and your competitor, given the decisions you both make in pricing the suit.

Let's find the Nash Equilibrium of this game using the "Best Response" method.

Competitor

		$50	$70
You	$50	[55%, 45%]	70%, 30%
	$70	[40%, 60%]	55%, 45%

Therefore, if you and your competitor's main goal is to sell as many suits as possible, the best price to sell the suit at is $50 for both of you. Notice that you or your competitor cannot sell more suits if either of you were to individually change the price of the suit.

(b) After doing research, you find out that 100 people will be buying this hot new suit. If you and your competitor's main goal is to maximize the amount of money made from selling the suit, what is the best price to sell the suit at?

The first thing we have to do is find out how much money will be made depending on what price you and your competitor sell the suit as:

- If you both sell the suit at $50:

  You will make: 100 * 0.55 * $50 = $2,750
  Your competitor will make: 100 * 0.45 * $50 = $2250

- If you sell the suit at $50 and your competitor sells it at $70:

  You will make: 100 * 0.70 * $50 = $3500
  Your competitor will make: 100 * 0.30 * $70 = $2100

- If you sell the suit at $70 and your competitor sells it at $50:

  You will make: 100 * 0.40 * $70 = $2800

  Your competitor will make: 100 * 0.60 * $50 = $3000

- If you both sell the suit at $70:

  You will make: 100 * 0.55 * $70 = $3850

  Your competitor will make: 100 * 0.45 * $70 = $3150

Now we can make our new game:

Competitor

		$50	$70
You	$50	$2750, $2250	$3500, $2100
	$70	$2800, $3000	$3850, $3150

Let's find the Nash Equilibrium of this game:

Competitor

		$50	$70
You	$50	[2750, 2250]	3500, 2100
	$70	2800, 3000	[3850, 3150]

Therefore, if both you and your competitor's goal is to maximize the amount of money you make from selling the hot new suit, you both should sell the suit at $70.

More Types of Games

6. You are given 10 chocolate coins for you and a friend to share, however you get to decide how many chocolate coins each of you gets (you get 10 and your friend gets 0; you get 9 and your friend gets 1; etc.). After you decide, your friend can either accept the offer or decline it. If they decline it, you both get nothing. How would you split the chocolate coins?

   Answers may vary. There are two cases you must consider:

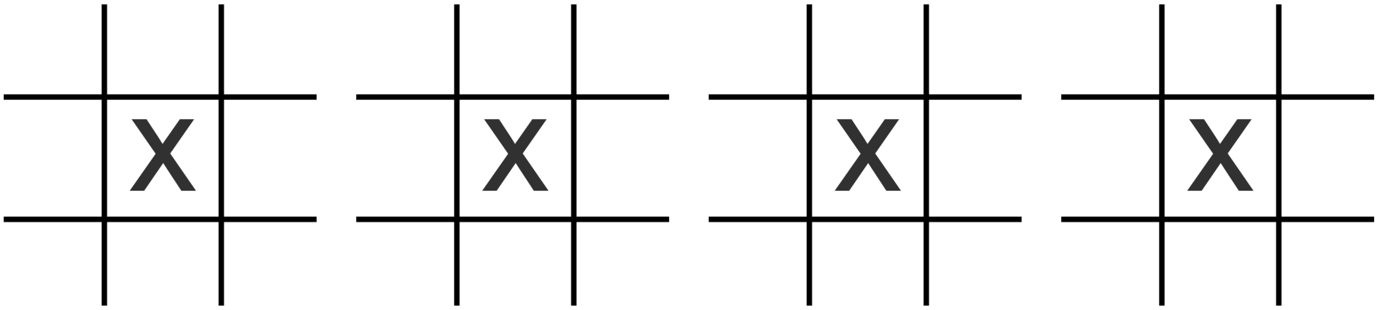
   (a) Your friend wants as many chocolate coins as possible.
   (b) Your friend wants you to share the coins fairly.

   If the first case is true, then you can give your friend only 1 chocolate coin and keep 9 for yourself, because you know they would rather have 1 chocolate coin than 0.

   If the second case is true, then you should give your friend 5 chocolate coins and keep 5 for yourself, because you know if you do not do this, then you will get 0 chocolate coins.

7. In the game tic-tac-toe, each player takes a turn to place an X or an O in one of nine spots. The player that forms a straight line of three X's or a straight line of three O's is the winner. The first player usually starts by placing an X in the middle spot on the grid.

   What are all of the spots player 2 can place an O next to give player 1 a winning strategy?



   Any spot but the corner spots .

8. **Twenty-One**

In the game twenty-one, you and a partner take turns subtracting numbers from 21 until one of you reaches zero. Each player can only subtract a number from 1 to 4 during each turn, and the person who reaches zero is the winner.

Play this game with a partner twice, alternating who goes first. Record your numbers in the tables below:

Game 1

You:																		
Partner:																		

Game 2

You:																		
Partner:																		

If you are the first player, there is a way you can win this game every time. What is this winning strategy?

(a) Subtract 1 as your first move.

(b) Choose a number that subtracts a total of 5 when combined with your partner's move each time.

9. * From the game above

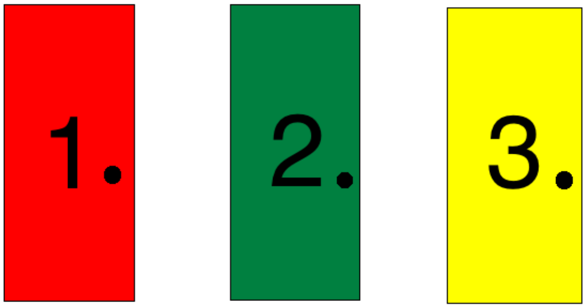
   (a) What would be the winning strategy if the game started at 24 instead of 21.
   Subtract 4 as your first move instead of 1, then repeat subtracting by 5.

   (b) Who would have the winning strategy if the game started at 20.
   The player with the second move.

   (c) If you are only allowed to subtract 1 and 2 each time, who has the winning strategy and what is it?
   The player with the second move has the winning strategy, and it would be to subtract a total of 3 each time when combined with the first player's move.

10. * Let's Make a Deal!

In the old game show Let's Make a Deal!, contestants are asked to choose between 3 doors. Behind one of the doors is a brand new car, but behind each of the other two doors is a goat. Once the contestant has chosen a door, one of the other two doors is opened, revealing a goat. The contestant is then given a choice to keep the door that he or she has chosen, or switch to the other remaining door. What should the contestant do?



Play this game with a partner. Repeat the game 6 times with you as the contestant (record your results on this sheet), and 6 times with your partner as the contestant (your partner will record his or her results on his or her sheet). Make sure you and your partner choose a different strategy that you will stick with for all 3 trials (ie. you choose to switch doors every time and your partner chooses to keep the same door everytime.)

Trial #	Door Picked	Door With Goat	Switch/Don't Switch	Win/Lose
1				
2				
3				
4				
5				
6				

What is the better strategy?

The better strategy is to switch doors every time, because it gives you a probability of $\frac{2}{3}$ of winning the car. This is because you originally only had a probability of $\frac{1}{3}$ of choosing the car to begin with, and a probability of $\frac{2}{3}$ that the car was behind one of the other two doors. Since we eliminate one of those other two doors, the probability of $\frac{2}{3}$ shifts to the remaining door.

11. * The Pirate Game

Five pirates were sailing one day and stumbled upon a treasure chest filled with 10 gold coins. The captain, Nash, and pirates 2, 3, 4, and 5 had to decide how the gold was to be shared. Being the captain, Nash was the first one to make a decision. The rules and conditions of the game are as follows:

Rules and Conditions:

- Nash is to propose how the pirates will share the gold, and the rest of them vote whether or not they agree. If Nash gets at least 50% of the vote in his favour, the gold will be shared his way (Nash's vote counts).
- If the vote is less than 50%, then Nash is thrown off the ship and pirate 2 becomes the captain and the game is repeated.
- The pirates' first goal is to remain on the ship and their second goal is to maximize the amount of gold coins they get.
- Assume that all 5 pirates are intelligent, rational, greedy, and do not wish to be thrown off the ship.

Nash finds a plan to maximize his gold and stay alive. What is the plan? (Hint: work backwards from the situation if it was just pirates 4 and 5 on the ship. What's the offer pirate 4 can make? What offer must pirate 5 accept before he is left alone with pirate 4?

Answers may vary, but there is the optimal solution:

The way we find the optimal solution is by working backwards to see if we can find any situations where a pirate will accept a low amount of gold. Remember, Nash wants to give away the least amount of gold coins as possible. If we can find these situations, then Nash, being an intelligent pirate, will also be able to find them.

Assume that pirates Nash, 2, and 3 have been thrown off the ship after their offers of how to share the gold were rejected. We are then left with pirates 4 and 5:

4	10
5	0

In this situation, pirate 4 can take all 10 gold coins for himself and still get 50% of the votes, since his vote counts. Therefore, pirate 5, being greedy, must do what is necessary to avoid this situation where its just him and pirate 4 on the ship. So, now, let's assume pirate 3 is back on the ship:

3	9
4	0
5	1

In this situation, pirate 3 can offer pirate 5 only 1 gold coin, because, being intelligent, he knows that if pirate 5 does not accept this offer then pirate 5 will get 0 gold coins when it is pirate 4's turn to make an offer. So pirate 5 will accept this offer, being rational, and pirate 3 will get at least 50% of the vote. However, pirate 4, being greedy, looks at this situation and realizes that he must do what is necessary to avoid it, or else he will end up with zero gold coins. So, now, let's assume pirate 2 is back on the ship:

2	9
3	0
4	1
5	0

In this situation, pirate 2 can offer pirate pirate 4 only 1 gold coin, because, being intelligent, he knows that if pirate 4 does not accept this offer then pirate 4 will get 0 gold coins when it is pirate 3's turn to make an offer. So pirate 4 will accept this offer, being rational, and pirate 2 will get at least 50% of the vote. However, pirates 3 and 5 look at this situation and realize that they must do what is necessary to avoid it, or else they will end up with zero gold coins. So, now, let's assume pirate Nash is back on the ship:

Nash	8
2	0
3	1
4	0
5	1

In this situation, pirate Nash can offer pirates 3 and 5 each one gold coin, because, being intelligent, he knows that if pirates 3 and 5 do not accept this offer then they will both get 0 gold coins when it is pirate 2's turn to make an offer. So pirates 3 and

5 will accept this offer, being rational, and pirate Nash will get at least 50% of the vote. Therefore, Nash should offer pirates 3 and 5 one gold coin each and keep eight for himself.