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Grade 7/8 Math Circles

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Sets

Sets

A **set** is a collection of **unique** objects i.e. no two objects can be the same. Objects that belong in a set are called **members** or **elements**. Elements of set can be anything you desire - numbers, animals, sport teams.

Representing Sets

There are a **two ways** to describe sets, we can either

1. **List out the elements, separated by commas, enclosed by curly brackets (“{” and “}”)**

   Often times, we use a **capital letter** to abbreviate the set we are referring to. The letter usually tries to stand for something.

   (a) W = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

   (b) V = {a, e, i, o, u}

   (c) C ={red, blue, yellow, green}

   (d) $N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

2. **Specify in words that objects in the set must have certain characteristics/properties**

   Describing elements using words proves useful when we are describing sets of large sizes (where listing every element proves cumbersome).

   (a) J = {All prime ministers of Canada}

   (b) C ={All past Math Circles sessions}

   (c) N = {Even numbers between 1 and 25}

   (d) P = {All prime numbers less than 100}

There are two important properties of sets you should be mindful of:

- **Elements are Unique** means elements appear only once in the set.

  That means $\{11, 6, 6\}$ should be written as $\{11, 6\}$

- **Elements are Unordered** means that the order of how you write the elements in a set does not matter. Two sets are considered the same so long as both sets contain the same elements.

  For example, $\{2, 5, 4, 3\}$ is the same as $\{4, 5, 3, 2\}$

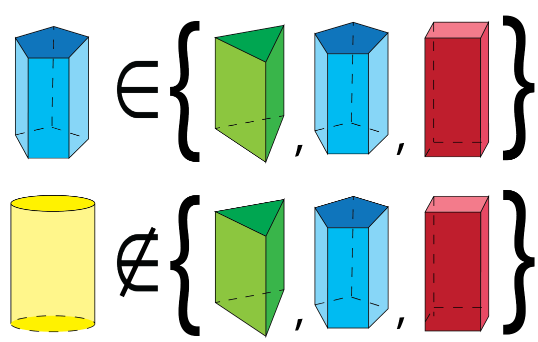
Exercise. Are the following statements true or false? If false, explain why:

1. Is $\{1, 1, 2, 3, 4, 5\}$ a valid set based on what we learned in this lesson?
2. Is the set $\{1, 3, 5, 7, 9\}$ the same as the set $\{1, 3, 5, 9, 7\}$?

Elements of a Set

When we want to show that something belongs in a set, we use the $\in$ symbol and we use $\notin$ to show that something does not belong in a set.

Example.



Universal Set and Empty Set

The **universal set** is all the elements that one wishes to consider in a situation. Any group of objects under examination is a universal set so long as we confine ourselves to just those objects. From the universal set, we can form sets.

Example. Suppose we only are examining quadrilaterals i.e. four sided figures (trapezoid, parallelogram, kite, rhombus, rectangle, square). This is our universal set.

We could have a set of just a rectangle and square. We can denote that as $\mathbb{S} = \{\text{rectangle}, \text{square}\}$

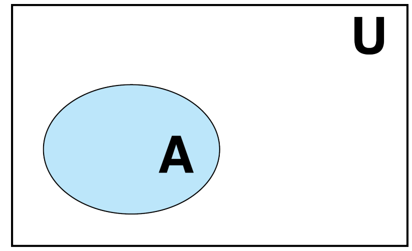


Figure 1: We use a rectangle to represent the universe $\mathcal{U}$ and circles to represent sets

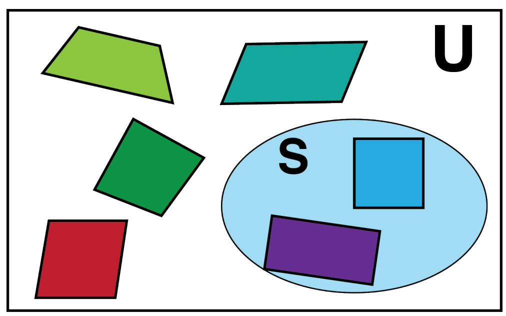


Figure 2: The objects in consideration are quadrilaterals

The Empty Set

Similarly, just as we can consider everything in the universal set, we could also consider nothing which we refer to as the **empty set**. It is the unique set with no elements.

We write the empty set as either $\{\}$ or $\emptyset$

Cardinality/Size of a Set

The number of elements/members in a set is called the **cardinality**. We place vertical bars around a set to indicate we want to find the cardinality of a set (think of it as the size of set).

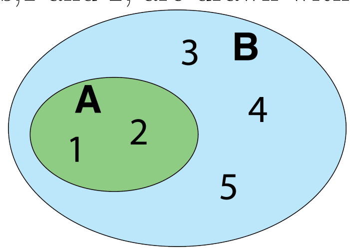
Example. Suppose we have $\mathbb{S} = \{561, 1105, 1729, 2465\}$. Then $|\mathbb{S}|$ has a cardinality of 4.

Example. The empty set has a cardinality of 0, since it has no elements.

Subsets and Supersets

Suppose we have two sets: $A = \{1, 2\}$ and $B = \{1, 2, 3, 4, 5\}$. Notice that every element of A is also an element of B. We say, then, that A is a **subset** of B. We write this as $A \subseteq B$ (pronounced A is contained in B). Equivalently, we could write $B \supseteq A$ (pronounced B is a **superset** of A).

Figure 3: The elements,1 and 2, are drawn within A but also within B



Exercises.

1. Given the set $S = \{2, 4\}$, determine if it is a subset of the following:
   (a) $\{2, 4, 16\}$
   (b) $\{1, 2, 4\}$
   (c) $\{1, 2\}$
   (d) $\{2, 4\}$
2. Consider the set $S = \{1, 2, 3, 4\}$
   (a) How many 3-element subsets are there?
   (b) How many 3-element subsets are there that must include 1 as an element?
   (c) In total how many possible subsets can there be?

Basic Set Operations

Just like how we can perform arithmetic operations (addition, subtraction, multiplication) with numbers, we can perform certain operations on sets. There are **three** fundamental operations we shall discuss:

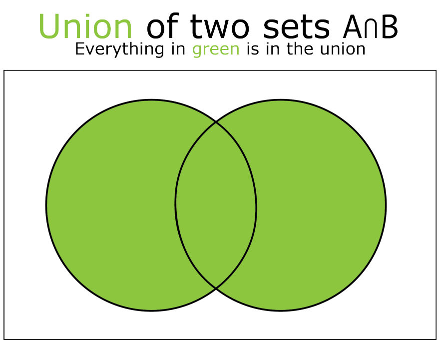
- Union
- Intersection
- Complement

Often times it is useful to use **Venn** diagrams to show how these operations work.

Union

The **union** includes elements from two sets while excluding any duplicates that may appear. In the union, every element belongs in either set A or B. The keyword is **"or"** which indicates that an element may belong in either A, B, or even possibly both. Mathematically, we write the union of two sets A and B as:

$$A \cup B$$



Visually, it is the area of both circles

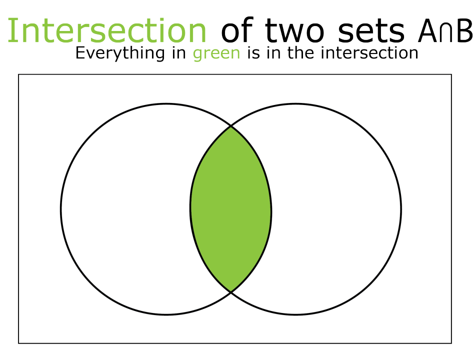
Example. Evaluate the following:

1. $\{1, 2\} \cup \{2, 3\}$
2. $\{1, 2\} \cup \{3, 4\}$
3. $\{1, 2\} \cup \{1, 2\}$
4. $\{1\} \cup \{1, 2, 3\}$

Intersection

The **intersection** is a new set formed from the elements common to two (or more) sets, A and B. The keyword is **"and"** indicating that elements must be in both in A and B. We write the intersection of A and B as

$$A \cap B$$



Visually, it is the area of where the two circles overlap

While the union is the set of all the elements in A and B, the intersection is the set that includes elements that are in A but also in B. In some cases, there are no common elements and the intersection is the empty set.

Examples. Evaluate the following

1. $\{1, 2\} \cap \{2, 3\}$
2. $\{1, 2\} \cap \{3\}$
3. $\{1, 2\} \cap \{1, 2\}$

Complement

When we are dealing with sets found within the universal set $\mathbb{U}$, we can define the complement of any set. The complement of the set A is defined as all the elements that are not in A with respect to the universal set. We show this mathematically by placing a bar over the set.

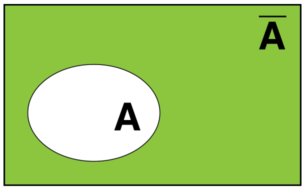


Figure 4: Everything outside the circle representing the set A, is the complement, $\bar{A}$

Example. Given $\mathbb{U} = \{1, 2, 3, 4, 5\}$ and $S = \{1, 2\}$

The complement of S, $\bar{S}$, is $\{3, 4, 5\}$.

Example. The complement of $\mathbb{U}$ is $\emptyset$ i.e the empty set.

Exercises.

1. Let A = { 1, 3, 5, 7, 9 }, B = { 2, 4, 6, 8, 10 } and C = { 2, 3, 5, 7, 11 }. Write out the following sets if $\mathbb{U}$ = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}:
   (a) $A \cup C$
   (b) $\overline{A}$
   (c) $(A \cap C) \cup B$
   (d) $\overline{B \cap C}$

Principle of Inclusion and Exclusion

Counting the number of elements for a single set is a relatively easy. However, we need a bit more care when we are determining the number of elements in the union of two or more sets.

Example. Consider the following two sets:
$T = \{1, 3, 6, 10, 15, 21, 28, 36\}$ and $S = \{1, 4, 9, 16, 25, 36\}$

Determine

1. $|T|$ and $|S|$
2. $|S \cup T|$

1. We can see that the cardinality of $|T|$ and $|S|$ is 8 and 6 respectively.

2. However, to find $|T \cup S|$, we can't just add $|T|$ and $|S|$ together and get $8 + 6 = 14$. This is because we are double counting elements common to both S and T i.e. $|S \cap T|$.

Notice T includes elements that are exclusive to just T but also elements common to both T and S i.e. $T \cap S$. Likewise, S has elements exclusive to just S but also elements common to S and T. So when we add $|T|$ and $|S|$, we are adding elements exclusive to T and S, but elements common to both S and T twice!

To compensate for this, we can subtract the intersection from $|T| + |S|$ leading us to the formula:

$$|S \cup T| = |S| + |T| - |S \cap T|$$

First let's determine $S \cap T = \{1, 36\}$ and hence $|S \cap T| = 2$

Applying the formula above, we get

$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ &= 6 + 8 - 2 \\ &= 12 \end{aligned}$$

We can check by determining $T \cup S = \{1, 3, 4, 6, 9, 10, 15, 16, 21, 25, 28, 36\}$, which after counting we can see has 12 elements.

Exercise.

1. How many 9-element subsets can be formed from the set $S = \{1, 2, ..., 15\}$ with the following conditions
    (a) contain 1 or 2
    (b) contain both 1 and 2 or 2 and 3
2. In the set $S = \{1, 2, 3, ...100\}$, how many elements in the set are divisible by both 6 and 7. Write out the final set of all elements.
3. There are 140 first year university students attending math. 52 have signed up for algebra, 71 for calculus, and 40 for statistics. There are 15 students who have take both calculus and algebra, 8 who are taking calculus and statistics, 11 who are taking algebra and statistics and 2 students are taking all three subjects.
    (a) How many students have not yet signed up for any courses? i.e. they are not taking any courses
    (b) Illustrate your with a Venn diagram

Infinite Sets

So far we have been dealing with sets that have a finite amount of elements. However, sets can also have a cardinality of infinity, that is a set can have an infinite number of elements.

When dealing with infinite sets, we often use ... right before the end of the right curly bracket }. This lets us know that the set continues on forever.

Example. We can write the set of all even numbers as follows

$$\mathcal{E} = \{2, 4, 6, 8, ...\}$$

Common Sets

There are certain infinite sets (which you may have already seen) that has specific name and symbol associated with them

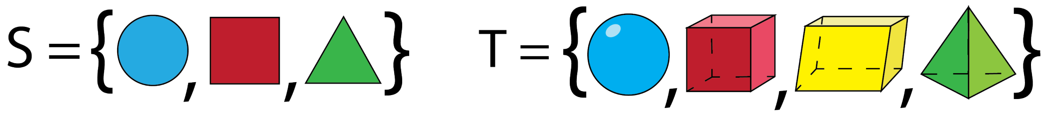
1. $\mathbb{N}$ - the set of all **natural** numbers. These are whole numbers you use everyday for counting and ordering
$$\{1, 2, 3, 4, 5, ...\}$$
2. $\mathbb{Z}$ - the set of all **integers**. These include both positive and negative numbers but no fractional component
$$\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$
3. $\mathbb{Q}$ - the set of all **rationals** (all possible fractions).
4. $\bar{\mathbb{Q}}$ - the set of all **irrationals** (all infinite non-repeating decimal numbers such as π
5. $\mathbb{R}$ - the set of all **real** numbers (includes both irrational and rational numbers)

When we are dealing with infinite sets, the rules that we learned for cardinality previously do not necessarily apply.

Finding a Match Between Two Sets

Example. Suppose we have two sets S and T of consisting of two dimensional and three dimesional shapes respectively. Determine if these two sets have the same cardinality.

We can easily see that these two sets do not have the same cardinality. We could count the number of elements in set S and T and conclude that these two sets are in fact not the same size.



However, there is an even more fundamental way to determine if these two sets are of the same size. We could simply pair one member of one set to a member of another set. If there are any remaining members that can't be paired up, then the set with remaining elements has more elements than the other.

In the above example, notice that we can pair a 2-D dimensional in set S to a 3-D dimensional shape exactly three times leaving the parallelepiped (yellow shape) left over in set T. Because there is an element left unpaired in T, we can conclude that T has a greater cardinality than S.

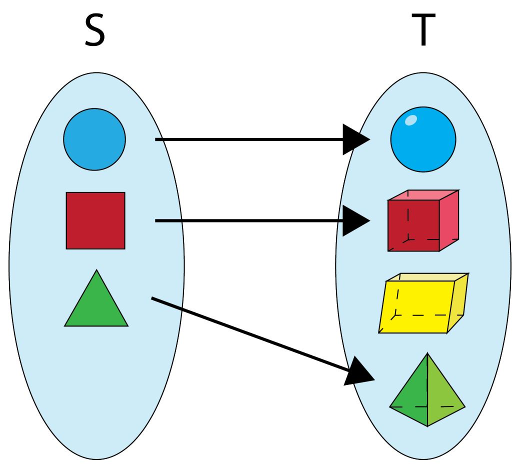


Figure 5: When matching S to an element in T, an element in T is left unpaired indicating that the cardinality of T is greater then S

This idea of matching allows us to determine if two sets are of the same size relatively quickly.

Example. The Lecture Room

In a lecture hall, a professor doesn't necessarily need to call out name by name to determine if he has full attendance. He simply needs to determine if all the chairs are filled.

The professor simply needs to match one chair to one student. If every chair is paired up, he knows that there are the same number of students as there are chairs even though he might not know how many of there of either.

Likewise, if there are any empty chairs, he knows that there are students who did not attend the lecture.

Counting to Infinity We will extend this idea of pairing two objects from different sets to sets with infinite cardinality.

Exercise. Here is something very puzzling, do the set of even numbers, $\mathcal{E} = \{2, 4, 6, 8, ...\}$ have the same cardinality as the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, 6, ...\}$?

You might be inclined to say because the even numbers are just part of the whole numbers that there are more natural numbers than even numbers. This type of reasoning only works if sets we are comparing are **finite** in size.

We resort to matching to see if two sets of the same size, since we are unable to count to infinity. We could go on forever, but we will never reach the end.

Let's pair up every natural number with it's double. i.e. 1 pairs up with 2, 2 pairs up with 4, 3 pairs up with 6....

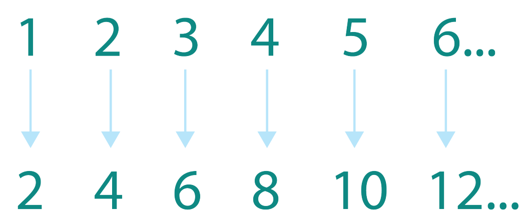


Figure 6: We can pair every natural number to an even number, by matching any natural number to it's double

Notice that by abiding by the above rule, we have matched every natural number to an even number and there are no elements left over from either sets. We are then forced to come to this astonishing conclusion: **there are as many even numbers as there are whole numbers!**

Example. Consider the set of even numbers $\mathcal{E} = \{2, 4, 6, 8, ...\}$ and the set of odd numbers $\mathcal{O} = \{1, 3, 5, 7, 9..\}$. Show that these two sets have the same cardinality by giving a rule that allows us to pair every even number to an odd number.

Definition 1 (**Countably Infinite**). *A set is **countably infinite** if its elements can be paired up with the set of natural numbers under some rule.*

Using the definition above, since we can match every even (or odd) number to the set of natural numbers, $\mathbb{N}$, we say that the set of even (or odd) numbers is countably infinite.

Hilbert's Grand Hotel

Hilbert's Hotel was a thought experiment which illustrates the strange and seemingly counter-intuitive results of infinite sets.

Example. Imagine a hotel with an infinite number of rooms numbered using the natural numbers i.e. 1, 2, 3, 4, 5, ...and so forth and a very hard working manager. Suppose one night, the infinite hotel room was booked up with an infinite numbers of guests and a man walks into the hotel and asks for a room.

How would the manager accommodate the man?

Exercise. Now let's suppose we have an infinite amount of people coming off from a bus requesting for a room in Hilbert's Hotel. In this case, how would we move everyone in the hotel to accommodate everyone coming off the bus. Can you ensure that there are enough rooms for everyone in the bus?

Problem Set

1. Determine the following

   (a) $\{1, 2, 3, 4\} \cup \{5, 6\}$

   (b) $\{1, 8, 27\} \cap \{9, 18, 27\}$

   (c) $\{\text{Integers greater than } 4\} \cap \{\text{Integers less than } 6\}$

   (d) $\{\text{Multiples of } 3\} \cap \{\text{Multiples of } 4\}$

   (e) $\{\text{Factors of } 100\} \cap \{\text{Factors of } 20\}$

   (f) $\{\text{Even Numbers}\} \cap \{\text{Prime Numbers}\}$

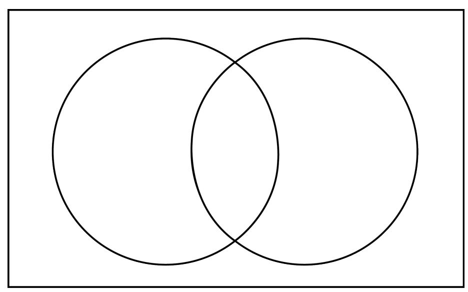
2. For the Venn Diagram, below, shade/identify the following regions

   (a) $A \cap B$

   (b) $\overline{A \cup B}$

   (c) $\bar{B}$

   (d) $A \cap \bar{A}$



3. Let $\mathbb{U}$ be the Universal Set and A be a set within the Universal Set, simplify the following

   (a) $\overline{\mathbb{U}}$

   (b) $A \cap \bar{A}$

   (c) $A \cup \bar{A}$

   (d) $\overline{\bar{A}}$?

   (e) $\{\ \} \cap \{\ \}$

   (f) $(A \cap B) \cup (A \cap \bar{B})$

4. Which of the following sets is a universal set for the other four sets

   (a) The set of even natural numbers

   (b) The set of odd natural numbers

   (c) The set of natural numbers

   (d) The set of negative numbers

(e) The set of integers

5. Find the number of subsets for

   (a) 1-element set

   (b) 2-element set

   (c) 3-element set

   (d) Did you notice a pattern regarding the number of subsets? Can you guess the number of subsets for an n-element set?

6. Determine the number of sets X such that $\{1, 2, 3\} \subset X \subset \{1, 2, 3, 4, 5, 6, 7\}$.
   Hint: What is the fewest possible elements X can have? What is the most elements X can have?

7. Set A comprises all three digit numbers that are multiples of 5, Set B comprises all three digit even numbers that are multiples of 3. How many elements are present in $A \cup B$?

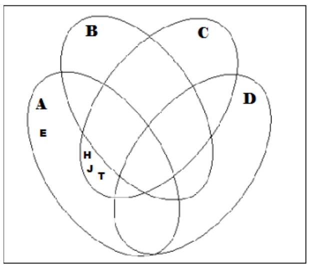
8. The Canadian Embassy in the United States has 30 people stationed there. 22 of the employees speak French and 15 speak English. If there are 10 who speak both French and English, how many of the employees speak

   (a) English

   (b) French

   (c) Neither French nor English

9. Let $A = \{E, H, J, T, S, G, Z, N\}$
   $B = \{L, Y, U, I, O, Z, N, R, K\}$,
   $C = \{G, Q, N, R, K, Y, F, Z, S, H, J, T\}$ and
   $D = \{U, I, N, G, F, X, R, Z, K, V\}$:



   (a) Redraw the 4-set Venn Diagram shown to the right, and place all the elements in their proper section.

   (b) How many of the sections in the diagram to the right represent the intersection of 2 sets? 3 sets? All 4 sets?

   (c) What elements are in the set $(A \cap B) \cup (C \cap D)$? Shade this in on a Venn Diagram.

(d) What elements are in the set $(\overline{B} \cap \overline{C}) \cap (A \cup D)$? Shade this in on a Venn Diagram.

10. **Hilbert's Hotel** Referring the section where we explored Hilbert's hotel, suppose we have an infinite number of buses with infinite number of passengers wanting a room. How would the hotel manager find room for everyone?

11. Is the size of the set {Numbers from 0 to 1} equal to the size of the set $\{-\infty$ to $\infty\}$? (Hint: Remember that there are decimal numbers between 0 and 1)

12. Write the Inclusion-Exclusion Principle for $|A \cup B \cup C \cup D \cup E|$.