



Grade 7/8 Math Circles

February 21st/22nd, 2017

Sets

Sets

A **set** is a collection of **unique** objects i.e. no two objects can be the same. Objects that belong in a set are called **members** or **elements**. Elements of set can be anything you desire - numbers, animals, sport teams.

Representing Sets

There are a **two ways** to describe sets, we can either

1. **List out the elements, separated by commas, enclosed by curly brackets (“{” and “}”)**

Often times, we use a **capital letter** to abbreviate the set we are referring to. The letter usually tries to stand for something.

- (a) $W = \{\text{Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}\}$
- (b) $V = \{\text{a, e, i, o, u}\}$
- (c) $C = \{\text{red, blue, yellow, green}\}$
- (d) $N = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

2. **Specify in words that objects in the set must have certain characteristics/properties**

Describing elements using words proves useful when we are describing sets of large sizes (where listing every element proves cumbersome).

- (a) $J = \{\text{All prime ministers of Canada}\}$
- (b) $C = \{\text{All past Math Circles sessions}\}$
- (c) $N = \{\text{Even numbers between 1 and 25}\}$
- (d) $P = \{\text{All prime numbers less than 100}\}$

There are two important properties of sets you should be mindful of:

- **Elements are Unique** means elements appear only once in the set.

That means $\{11, 6, 6\}$ should be written as $\{11, 6\}$

- **Elements are Unordered** means that the order of how you write the elements in a set does not matter. Two sets are considered the same so long as both sets contain the same elements.

For example, $\{2, 5, 4, 3\}$ is the same as $\{4, 5, 3, 2\}$

Exercise. Are the following statements true or false? If false, explain why:

1. Is $\{1, 1, 2, 3, 4, 5\}$ a valid set based on what we learned in this lesson?

Solution. No, since every element in a set is unique.

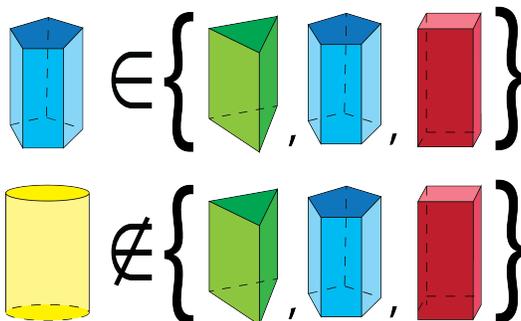
2. Is the set $\{1, 3, 5, 7, 9\}$ the same as the set $\{1, 3, 5, 9, 7\}$?

Solution. Yes, the two sets have the same elements but arranged in a different order.

Elements of a Set

When we want to show that something belongs in a set, we use the \in symbol and we use \notin to show that something does not belong in a set.

Example.



Universal Set and Empty Set

The **universal set** is all the elements that one wishes to consider in a situation. Any group of objects under examination is a universal set so long as we confine ourselves to just those objects. From the universal set, we can form sets.

Example. Suppose we only are examining quadrilaterals i.e. four sided figures (trapezoid, parallelogram, kite, rhombus, rectangle, square). This is our universal set.

We could have a set of just a rectangle and square. We can denote that as $S = \{\text{rectangle, square}\}$

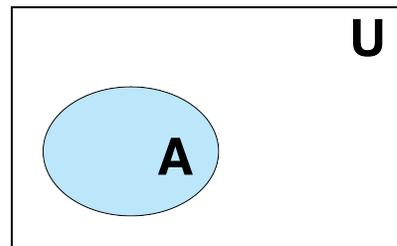


Figure 1: We use a rectangle to represent the universe \mathcal{U} and circles to represent sets

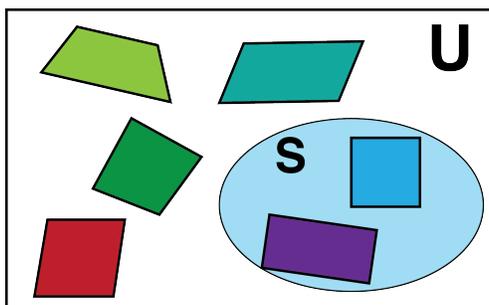


Figure 2: The objects in consideration are quadrilaterals

The Empty Set

Similarly, just as we can consider everything in the universal set, we could also consider nothing which we refer to as the **empty set**. It is the unique set with no elements.

We write the empty set as either $\{\}$ or \emptyset

Cardinality/Size of a Set

The number of elements/members in a set is called the **cardinality**. We place vertical bars around a set to indicate we want to find the cardinality of a set (think of it as the size of set).

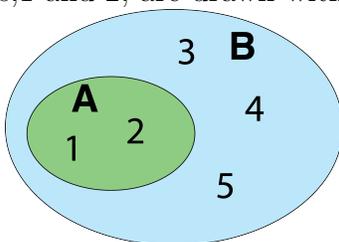
Example. Suppose we have $S = \{561, 1105, 1729, 2465\}$. Then $|S|$ has a cardinality of 4.

Example. The empty set has a cardinality of 0, since it has no elements.

Subsets and Supersets

Suppose we have two sets: $A = \{1, 2\}$ and $B = \{1, 2, 3, 4, 5\}$. Notice that every element of A is also an element of B. We say, then, that A is a **subset** of B. We write this as $A \subseteq B$ (pronounced A is contained in B). Equivalently, we could write $B \supseteq A$ (pronounced B is a **superset** of A).

Figure 3: The elements, 1 and 2, are drawn within A but also within B



Exercises.

1. Given the set $S = \{2, 4\}$, determine if it is a subset of the following:

(a) $\{2, 4, 16\}$

Solution. Yes

(b) $\{1, 2, 4\}$

Solution. Yes

(c) $\{1, 2\}$

Solution. No, since $4 \notin \{1, 2\}$

(d) $\{2, 4\}$

Solution. Yes, even though they are the same set.

2. Consider the set $S = \{1, 2, 3, 4\}$

(a) How many 3-element subsets are there?

Solution. We can choose any 3 of the 4 elements to compose a 3-element subset. Recall that from a past math circles session, that 4 choose 3 is calculated as follows:

$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{4}{1} = 4$$

Alternatively, students may choose to exhaust all possible 3-element subsets:

$$\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}$$

∴ There are 4 different possible 3-element subsets.

(b) How many 3-element subsets are there that must include 1 as an element?

Solution. Since 1 must be included in the subset, that leaves 2 elements to choose out of the 3 remaining possibilities, 2, 3, 4. Hence:

$$\binom{3}{2} = \frac{3!}{2!(2-1)!} = \frac{3!}{2!} = \frac{3}{1} = 3$$

Alternatively, students may choose to find all possible 3-element subsets, and then find the all the subsets with 1 included.

$$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}$$

∴ There are 3 3-element subsets that must include 1.

(c) In total how many possible subsets can there be?

Solution. There are in total (including the empty set):

$$\begin{aligned} & \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} \\ &= \frac{4!}{0!(4-0)!} + \frac{4!}{1!(4-1)!} + \frac{4!}{2!(4-2)!} + \frac{4!}{3!(4-3)!} + \frac{4!}{4!(4-4)!} \\ &= \frac{4!}{0!4!} + \frac{4!}{1!3!} + \frac{4!}{2!2!} + \frac{4!}{3!1!} + \frac{4!}{4!0!} \\ &= 1 + 4 + 6 + 4 + 1 \\ &= 16 \end{aligned}$$

∴ There are 16 different possible subsets that can be formed from $S = \{1, 2, 3, 4\}$

Basic Set Operations

Just like how we can perform arithmetic operations (addition, subtraction, multiplication) with numbers, we can perform certain operations on sets. There are **three** fundamental operations we shall discuss:

- Union
- Intersection
- Complement

Often times it is useful to use **Venn** diagrams to show how these operations work.

Union

The **union** includes elements from two sets while excluding any duplicates that may appear. In the union, every element belongs in either set A or B . The keyword is “**or**” which indicates that an element may belong in either A , B , or even possibly both. Mathematically, we write the union of two sets A and B as:

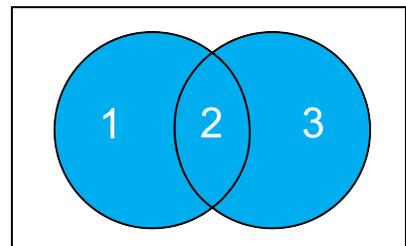
$$A \cup B$$

Example. Evaluate the following:

1. $\{1, 2\} \cup \{2, 3\}$
2. $\{1, 2\} \cup \{3, 4\}$
3. $\{1, 2\} \cup \{1, 2\}$
4. $\{1\} \cup \{1, 2, 3\}$

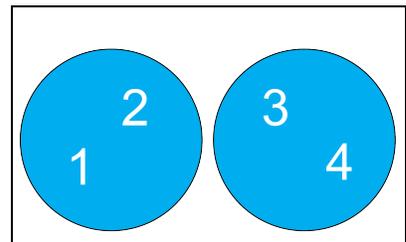
1. **Solution.** $\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\}$

We simply write down all the elements found in both sets. Notice that “2” is common to both sets. Due to the **uniqueness property**, every element in a set appears once, so the union should include 2 only once.



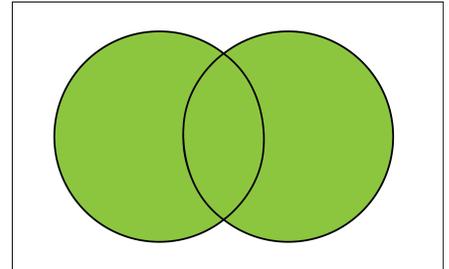
2. **Solution.** $\{1, 2\} \cup \{3, 4\} = \{1, 2, 3, 4\}$

There are no elements common to both sets, so we include all elements. Because there are no common elements, there should not be an overlapping region in our Venn diagram and as such the sets are drawn separate from other.



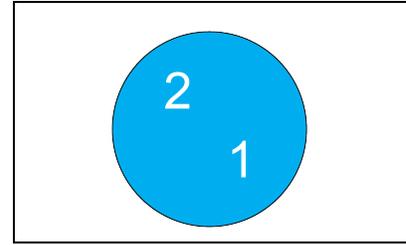
3. **Solution.** $\{1, 2\} \cup \{1, 2\} = \{1, 2\}$

Union of two sets $A \cap B$
Everything in green is in the union



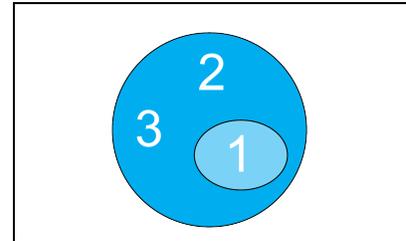
Visually, it is the area of both circles

We have two identical sets, but due to the uniqueness property, each element appears once in the union. Because the two sets are identical, the Venn diagrams are drawn on top of each other.



4. **Solution.** $\{1\} \cup \{1, 2, 3\} = \{1, 2, 3\}$

In this particular example, $\{1\}$ is a subset of $\{1, 2, 3\}$. Due to the uniqueness property 1 only appears once in the union. In our Venn diagram, the set $\{1\}$ is drawn within $\{1, 2, 3\}$.



Intersection

The **intersection** is a new set formed from the elements common to two (or more) sets, A and B . The keyword is “**and**” indicating that elements must be in both in A and B . We write the intersection of A and B as

$$A \cap B$$

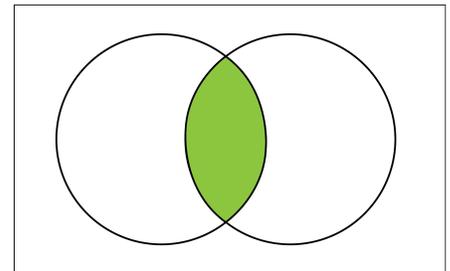
While the union is the set of all the elements in A and B , the intersection is the set that includes elements that are in A but also in B . In some cases, there are no common elements and the intersection is the empty set.

Examples. Evaluate the following

1. $\{1, 2\} \cap \{2, 3\}$
2. $\{1, 2\} \cap \{3\}$
3. $\{1, 2\} \cap \{1, 2\}$

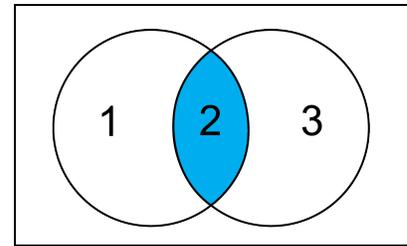
1. $\{1, 2\} \cap \{2, 3\}$

Intersection of two sets $A \cap B$
 Everything in green is in the intersection



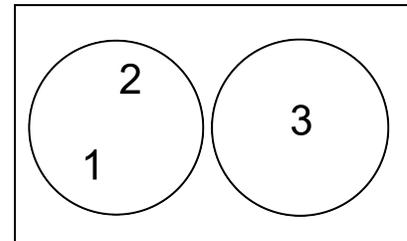
Visually, it is the area of where the two circles overlap

Solution. There is only one element common to both sets notably 2. The intersection thus only contains 2. On the Venn diagrams, it is the overlapping (or shared) region of the two sets.



2. $\{1, 2\} \cap \{3\}$

Solution. There are no elements common to both sets. Hence the intersection is the empty set i.e. nothing. In addition, the Venn diagrams should be drawn separate due to the lack of shared elements.



Complement

When we are dealing with sets found within the universal set \mathbb{U} , we can define the complement of any set. The complement of the set A is defined as all the elements that are not in A with respect to the universal set. We show this mathematically by placing a bar over the set.

Example. Given $\mathbb{U} = \{1, 2, 3, 4, 5\}$ and $S = \{1, 2\}$

The complement of S , \bar{S} , is $\{3, 4, 5\}$.

Example. The complement of \mathbb{U} is \emptyset i.e the empty set.

Exercises.

1. Let $A = \{1, 3, 5, 7, 9\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{2, 3, 5, 7, 11\}$. Write out the following sets if $\mathbb{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$:

(a) $A \cup C$

Solution. $A \cup C = \{1, 2, 3, 5, 7, 9, 11\}$

(b) \bar{A}

Solution. $\{2, 4, 6, 8, 10, 11\}$

(c) $(A \cap C) \cup B$

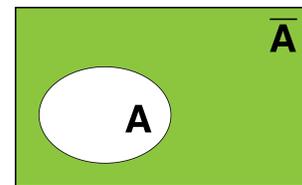


Figure 4: Everything outside the circle representing the set A , is the complement, \bar{A}

Solution.

$$\begin{aligned}(A \cap C) \cup B &= (\{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7, 11\}) \cup \{2, 4, 6, 8, 10\} \\ &= \{3, 5, 7\} \cup \{2, 4, 6, 8, 10\} \\ &= \{2, 3, 5, 6, 7, 8, 10\}\end{aligned}$$

(d) $\overline{B \cap C}$

Solution.

$$\begin{aligned}&= \overline{B \cap C} \\ &= \overline{\{2, 4, 6, 8, 10\} \cap \{2, 3, 5, 7, 11\}} \\ &= \overline{\{2\}} \\ &= \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11\}\end{aligned}$$

Principle of Inclusion and Exclusion

Counting the number of elements for a single set is a relatively easy. However, we need a bit more care when we are determining the number of elements in the union of two or more sets.

Example. Consider the following two sets:

$$T = \{1, 3, 6, 10, 15, 21, 28, 36\} \text{ and } S = \{1, 4, 9, 16, 25, 36\}$$

Determine

1. $|T|$ and $|S|$
2. $|S \cup T|$

1. We can see that the cardinality of $|T|$ and $|S|$ is 8 and 6 respectively.
2. However, to find $|T \cup S|$, we can't just add $|T|$ and $|S|$ together and get $8 + 6 = 14$. This is because we are double counting elements common to both S and T i.e. $|S \cap T|$.

Notice T includes elements that are exclusive to just T but also elements common to both T and S i.e. $T \cap S$. Likewise, S has elements exclusive to just S but also elements common

to S and T . So when we add $|T|$ and $|S|$, we are adding elements exclusive to T and S , but elements common to both S and T twice!

To compensate for this, we can subtract the intersection from $|T| + |S|$ leading us to the formula:

$$|S \cup T| = |S| + |T| - |S \cap T|$$

First let's determine $S \cap T = \{1, 36\}$ and hence $|S \cap T| = 2$

Applying the formula above, we get

$$\begin{aligned} |S \cup T| &= |S| + |T| - |S \cap T| \\ &= 6 + 8 - 2 \\ &= 12 \end{aligned}$$

We can check by determining $T \cup S = \{1, 3, 4, 6, 9, 10, 15, 16, 21, 25, 28, 36\}$, which after counting we can see has 12 elements.

Exercise.

1. How many 9-element subsets can be formed from the set $S = \{1, 2, \dots, 15\}$ with the following conditions
 - (a) contain 1 or 2

We have to use the principle of exclusion and inclusion. First we find the number of 9-element subsets that includes 1 and the number of 9 elements sets that include 2 in it. Lastly, we find 9 element subsets that include both 1 and 2.

9-element subsets that include 1:

$$\binom{14}{8} = \frac{14!}{8!6!} = 3003$$

Similarly the number of 9-element subsets that include 2:

$$\binom{14}{8} = \frac{14!}{8!6!} = 3003$$

If we include both 1 and 2 in a 9-element subset, then we have 13 elements to

choose for the remaining 7-elements left.

$$\binom{13}{7} = \frac{13!}{7! 6!} = 1716$$

By the principle of exclusion and inclusion:

$$\begin{aligned} |1 \text{ or } 2 \text{ included}| &= |1 \text{ included}| + |2 \text{ included}| - |1 \text{ and } 2 \text{ included}| \\ &= 3003 + 3003 - 1716 \\ &= 4290 \end{aligned}$$

(b) contain both 1 and 2 or 2 and 3

We need to find the number of 9 element subsets that include 1 and 2 and 9-element subset that includes 2 and 3, and then finally subtract the 9-element subsets that include 1,2, and 3.

9-element subsets that include 1 and 2:

$$\binom{13}{7} = \frac{13!}{7! 6!} = 1716$$

9-element subsets that includes 2 and 3:

$$\binom{13}{7} = \frac{13!}{7! 6!} = 1716$$

9-element subsets that include 1, 2, and 3

$$\binom{12}{6} = \frac{13!}{6! 7!} = 924$$

By the principle of exclusion and inclusion:

$$\begin{aligned} &1716 + 1716 - 924 \\ &= 2508 \end{aligned}$$

2. In the set $S = \{1, 2, 3, \dots, 100\}$, how many elements in the set are divisible by both 6 and 7. Write out the final set of all elements.

The number of elements that are divisible by 7:

$$100 \div 7 \approx 14$$

\therefore there are 14 elements that are divisible by 7.

The number of elements that are divisible by 6:

$$100 \div 6 \approx 16$$

\therefore there are 16 elements that are divisible by 6.

The number of elements that are divisible by 42 in other words divisible by both 6 and 7:

$$100 \div 42 \approx 2$$

\therefore there are 2 elements that are divisible by both 6 and 7.

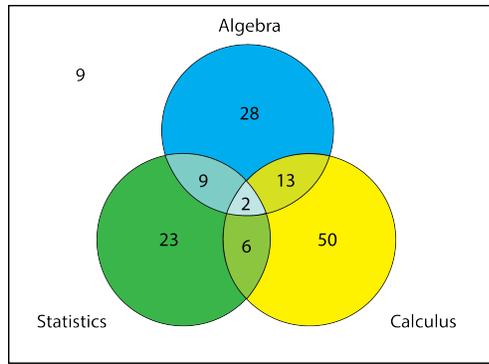
By the principle of inclusion and exclusion:

$$14 + 16 - 2 = 28$$

\therefore there are 28 elements in the set $\{1, 2, 3, \dots, 100\}$ that are divisible by 6 and 7.

3. There are 140 first year university students attending math. 52 have signed up for algebra, 71 for calculus, and 40 for statistics. There are 15 students who have take both calculus and algebra, 8 who are taking calculus and statistics, 11 who are taking algebra and statistics and 2 students are taking all three subjects.
- (a) How many students have not yet signed up for any courses? i.e. they are not taking any courses
 - (b) Illustrate your with a Venn diagram

We shall use a Venn diagram to represent the students taking algebra, calculus, and statistics courses. The overlapping region indicates they are taking two or more of these courses.



∴, from our Venn diagram, we can see that 9 people has not yet signed up for any courses.

Infinite Sets

So far we have been dealing with sets that have a finite amount of elements. However, sets can also have a cardinality of infinity, that is a set can have an infinite number of elements.

When dealing with infinite sets, we often use ... right before the end of the right curly bracket $\}$. This lets us know that the set continues on forever.

Example. We can write the set of all even numbers as follows

$$\mathcal{E} = \{2, 4, 6, 8, \dots\}$$

Common Sets

There are certain infinite sets (which you may have already seen) that has specific name and symbol associated with them

1. \mathbb{N} - the set of all **natural** numbers. These are whole numbers you use everyday for counting and ordering

$$\{1, 2, 3, 4, 5, \dots\}$$

2. \mathbb{Z} - the set of all **integers**. These include both positive and negative numbers but no fractional component

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

3. \mathbb{Q} - the set of all **rationals** (all possible fractions).

4. $\bar{\mathbb{Q}}$ - the set of all **irrationalals** (all infinite non-repeating decimal numbers such as π)

5. \mathbb{R} - the set of all **real** numbers (includes both irrational and rational numbers)

When we are dealing with infinite sets, the rules that we learned for cardinality previously do not necessarily apply.

Finding a Match Between Two Sets

Example. Suppose we have two sets S and T of consisting of two dimensional and three dimensional shapes respectively. Determine if these two sets have the same cardinality.

We can easily see that these two sets do not have the same cardinality. We could count the number of elements in set S and T and conclude that these two sets are in fact not the same size.

$$S = \{ \text{blue circle}, \text{red square}, \text{green triangle} \} \quad T = \{ \text{blue sphere}, \text{red cube}, \text{yellow parallelepiped}, \text{green pyramid} \}$$

However, there is an even more fundamental way to determine if these two sets are of the same size. We could simply pair one member of one set to a member of another set. If there are any remaining members that can't be paired up, then the set with remaining elements has more elements than the other.

In the above example, notice that we can pair a 2-D dimensional in set S to a 3-D dimensional shape exactly three times leaving the parallelepiped (yellow shape) left over in set T . Because there is an element left unpaired in T , we can conclude that T has a greater cardinality than S .

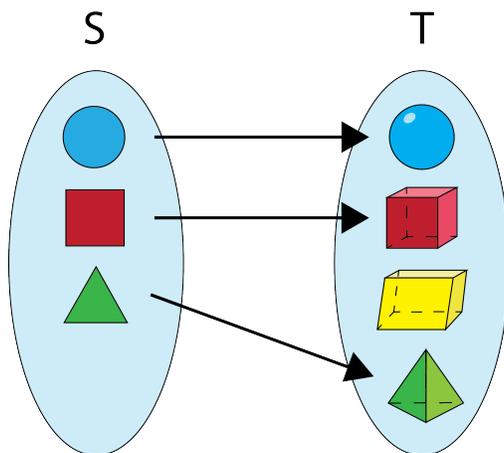


Figure 5: When matching S to an element in T , an element in T is left unpaired indicating that the cardinality of T is greater than S

This idea of matching allows us to determine if two sets are of the same size relatively quickly.

Example. The Lecture Room

In a lecture hall, a professor doesn't necessarily need to call out name by name to determine if he has full attendance. He simply needs to determine if all the chairs are filled.

The professor simply needs to match one chair to one student. If every chair is paired up, he knows that there are the same number of students as there are chairs even though he might not know how many of there of either.

Likewise, if there are any empty chairs, he knows that there are students who did not attend the lecture.

Counting to Infinity We will extend this idea of pairing two objects from different sets to sets with infinite cardinality.

Exercise. Here is something very puzzling, do the set of even numbers, $\mathcal{E} = \{2, 4, 6, 8, \dots\}$ have the same cardinality as the set of natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, 6, \dots\}$?

You might be inclined to say because the even numbers are just part of the whole numbers that there are more natural numbers than even numbers. This type of reasoning only works if sets we are comparing are **finite** in size.

We resort to matching to see if two sets of the same size, since we are unable to count to infinity. We could go on forever, but we will never reach the end.

Let's pair up every natural number with it's double. i.e. 1 pairs up with 2, 2 pairs up with 4, 3 pairs up with 6....

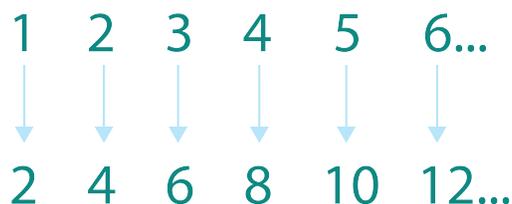


Figure 6: We can pair every natural number to an even number, by matching any natural number to it's double

Notice that by abiding by the above rule, we have matched every natural number to an even number and there are no elements left over from either sets. We are then forced to come to this astonishing conclusion: **there are as many even numbers as there are whole numbers!**

Example. Consider the set of even numbers $\mathcal{E} = \{2, 4, 6, 8, \dots\}$ and the set of odd numbers $\mathcal{O} = \{1, 3, 5, 7, 9, \dots\}$. Show that these two sets have the same cardinality by giving a rule that allows us to pair every even number to an odd number.

Solution. We can pair every even number to an odd number according to the following rule: take any even number and subtract 1 from it. This pairs all even numbers to all odd numbers.

Definition 1 (Countably Infinite). A set is *countably infinite* if its elements can be paired up with the set of natural numbers under some rule.

Using the definition above, since we can match every even (or odd) number to the set of natural numbers, \mathbb{N} , we say that the set of even (or odd) numbers is countably infinite.

Hilbert's Grand Hotel

Hilbert's Hotel was a thought experiment which illustrates the strange and seemingly counter-intuitive results of infinite sets.

Example. Imagine a hotel with an infinite number of rooms numbered using the natural numbers i.e. 1, 2, 3, 4, 5, ...and so forth and a very hard working manager. Suppose one night, the infinite hotel room was booked up with an infinite numbers of guests and a man walks into the hotel and asks for a room.

How would the manager accommodate the man?

Solution. He asks the guest in room 1 to move to room 2, the guest in room 2 to move to room 3, the guest in room 3 to move to room 4, and so forth. Every guest moves from their current room, n , to the room next, $n + 1$.

This eventually leaves room 1 open for the new man.

This process can be repeated for as many new customers as possible.

Exercise. Now let's suppose we have an infinite amount of people coming off from a bus requesting for a room in Hilbert's Hotel. In this case, how would we move everyone in the hotel to accommodate everyone coming off the bus. Can you ensure that there are enough rooms for everyone in the bus?

Solution. All we have to do is show that there are enough rooms for a countably infinite number of people coming off the bus. Remember, the number of people coming off the bus has the same size as the set of all odd numbers. This is key.

We will move everyone to a new room with a room number that is double their previous room number i.e. guest 1 will move from room 1 to room 2, guest 2 will move from room 2 to room 4, guest 3 will move from room 3 to room 6, and so forth.

Each guest in room n will move to room $2n$ this will fill up the infinite even number rooms, but it leaves all the INFINITE ODD number rooms open. We already know that the number of odd numbers has the same size as the number of people on the bus, even though they are both infinite.

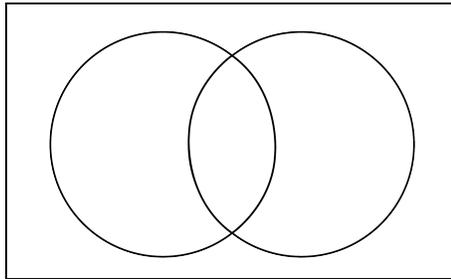
Problem Set

1. Determine the following

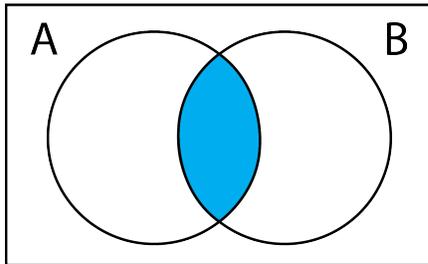
- (a) $\{1, 2, 3, 4\} \cup \{5, 6\}$
- (b) $\{1, 8, 27\} \cap \{9, 18, 27\}$
- (c) $\{\text{Integers greater than 4}\} \cap \{\text{Integers less than 6}\}$
- (d) $\{\text{Multiples of 3}\} \cap \{\text{Multiples of 4}\}$
- (e) $\{\text{Factors of 100}\} \cap \{\text{Factors of 20}\}$
- (f) $\{\text{Even Numbers}\} \cap \{\text{Prime Numbers}\}$

2. For the Venn Diagram, below, shade/identify the following regions

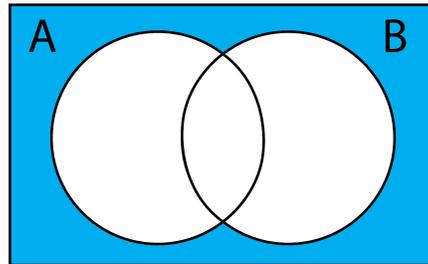
- (a) $A \cap B$
- (b) $\overline{A \cup B}$
- (c) \bar{B}
- (d) $A \cap \bar{A}$



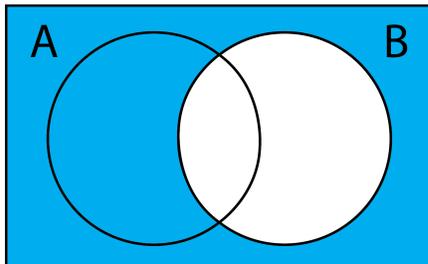
a)



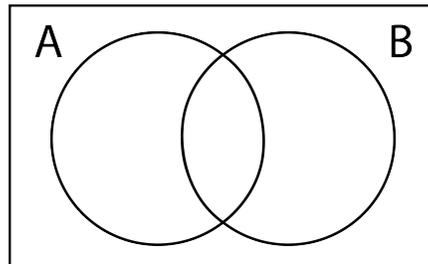
b)



c)



d)



3. Let U be the Universal Set and A be a set within the Universal Set, simplify the following

(a) \bar{U}

Solution. \emptyset

(b) $A \cap \bar{A}$

Solution. \emptyset

(c) $A \cup \bar{A}$

Solution. U

(d) $\bar{\bar{A}}$?

Solution. A

(e) $\{ \} \cap \{ \}$

Solution. \emptyset

(f) $(A \cap B) \cup (A \cap \bar{B})$

Solution. A

4. Which of the following sets is a universal set for the other four sets

(a) The set of even natural numbers

(b) The set of odd natural numbers

(c) The set of natural numbers

(d) The set of negative numbers

(e) The set of integers

Solution. The set of integers is the universal set for the other four sets

5. Find the number of subsets for

(a) 1-element set

Solution. There are **2** possible subsets, the empty set and the set containing itself

(b) 2-element set

Solution. There are **4** possible subsets. Let's represent the 2-element set as $\{a, b\}$, then the 4 possible subsets are: $\emptyset, \{a\}, \{b\}, \{a, b\}$

(c) 3-element set

Solution. There are **8** possible subsets. Let's represent the 3-element set as $\{a, b, c\}$, then the 8 possible subsets are: $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

- (d) Did you notice a pattern regarding the number of subsets? Can you guess the number of subsets for an n -element set?

Solution. For any n -element set, there are 2^n subsets

6. Determine the number of sets X such that $\{1, 2, 3\} \subset X \subset \{1, 2, 3, 4, 5, 6, 7\}$.

Hint: What is the fewest possible elements X can have? What is the most elements X can have?

All possible subsets of X must contain the required elements of 1, 2, and 3 and can have anywhere from 3-7 elements.

If X has 3-elements, then there is only 1 possible set, notably the set that contains just 1, 2, and 3. Subsets with a cardinality greater than 3, must have additional elements on top of the required elements of 1, 2, and 3 from 4 possible choices of 4,5,6, and 7.

If X had 4 elements, it would have 1 additional element

\therefore there are $\binom{4}{1} = 4$ possible sets

If X had 5 elements, it would have 2 additional elements

\therefore there are $\binom{4}{2} = 6$ possible sets

If X had 6 elements, it would have 3 additional elements

\therefore there are $\binom{4}{3} = 4$ possible sets

If X had 7 elements, it would have 4 additional elements

\therefore there are $\binom{4}{4} = 1$ possible sets

Therefore there are $1 + 4 + 6 + 4 + 1 = 16$ possible sets for X

7. Set A comprises all three digit numbers that are multiples of 5, Set B comprises all three digit even numbers that are multiples of 3. How many elements are present in $A \cup B$?

We shall denote $\lfloor \]$ as rounding a number down to the nearest whole number. For example, if $\lfloor 4.8 \rfloor = 4$

There are $\lfloor \frac{999}{5} \rfloor = 199$ numbers between 1 and 999 that are multiples of 5. There are $\lfloor \frac{99}{5} \rfloor = 19$ numbers between 1 and 99 that are multiples of 5. If we subtract 19 from 199, we get 180 3 digit numbers that are multiples of 5.

$\therefore |A| = 180$

Another way of looking at it, is that there are $5 \times 20, 5 \times 21, 5 \times 22, 5 \times 23 \times 5$, and so on to 5×199 . Therefore there are $199 - 20 + 1 = 180$ 3 digit numbers that are multiples of 5.

Applying a similar logic to set B. There are $\lfloor \frac{999}{3} \rfloor = 333$ numbers between 1 and 999 that are multiples of 3. There are $\lfloor \frac{99}{3} \rfloor = 33$ numbers between 1 and 99 that are multiples of 3. If we subtract 33 from 333, we get 300 3 digit numbers that are multiples of 3.

We can see it using a different way, we can see $3 \times 34, 3 \times 35, \dots$ and so forth until we reach 3×333 . Therefore there are $333 - 34 + 1 = 300$ 3 digit numbers that are multiples of 3.

$$\therefore |B| = 300$$

Now, we need to figure out how many 3 digit that are divisible by both 3 and 5 i.e. the intersection of 3 and 5. Numbers that are both multiples of 3 and 5 are multiples of 15.

Using the same technique as before, there are 60 3 digit numbers that are multiples of 15.

$$\therefore |A \cap B| = 60.$$

Using the principle of inclusion and exclusion, we have that

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 180 + 300 - 60 \\ &= 420 \end{aligned}$$

\therefore there are 420 numbers in $|A \cup B|$

8. The Canadian Embassy in the United States has 30 people stationed there. 22 of the employees speak French and 15 speak English. If there are 10 who speak both French and English, how many of the employees speak

(a) English

Since 10 who speak French and English, there are $15 - 10 = 5$ who speak English only

(b) French

Solution. Similarly, there are $22 - 10 = 12$ who speak French only.

(c) Neither French nor English

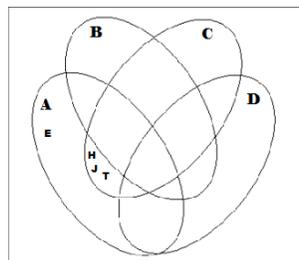
Solution. We first need to determine the number of people who speak French or English

Let F be the set of all people who speak French
 Let E be the set of all people who speak English
 Then $F \cup E$ is the set of people who speak English and French
 Using the Principle of Inclusion and Exclusion, we have

$$\begin{aligned} |F \cup E| &= |E| + |F| - |E \cap F| \\ &= 22 + 15 - 10 \\ &= 27 \end{aligned}$$

\therefore , we have 27 people who can speak either French or English. That leaves $30 - 27 = 3$ people in the Canadian Embassy who speaks neither French nor English.

9. Let $A = \{E, H, J, T, S, G, Z, N\}$
 $B = \{L, Y, U, I, O, Z, N, R, K\}$,
 $C = \{G, Q, N, R, K, Y, F, Z, S, H, J, T\}$ and
 $D = \{U, I, N, G, F, X, R, Z, K, V\}$:



- (a) Redraw the 4-set Venn Diagram shown to the right, and place all the elements in their proper section.
- (b) How many of the sections in the diagram to the right represent the intersection of 2 sets? 3 sets? All 4 sets?
 2 sets: 6 representations
 3 sets: 4 representations
 1 set: 1 representation
- (c) What elements are in the set $(A \cap B) \cup (C \cap D)$? Shade this in on a Venn Diagram.
 $(A \cap B) \cup (C \cap D) = \{F\}$
- (d) What elements are in the set $(\overline{B} \cap \overline{C}) \cap (A \cup D)$? Shade this in on a Venn Diagram.
 $(\overline{B} \cap \overline{C}) \cap (A \cup D) = \{F, Z, C, N\}$
10. **Hilbert's Hotel** Referring the section where we explored Hilbert's hotel, suppose we have an infinite number of buses with infinite number of passengers wanting a room. How would the hotel manager find room for everyone?

We will take advantage of Euclid's Lemma that asserts that there are an infinite number of prime numbers. The infinite number of prime numbers

Begin by assigning every person in room n , to a new room 2^n . For example, if a person is in the 8th room, they are now moved to room $2^8 = 256$.

He then assigns everyone on the 1st bus with infinite passengers, to the room 3^n where n is their seat number i.e. if the person was on the 6th seat on the bus, they now move in $3^6 = 729$ room.

The passengers on the second bus are assigned to powers of the next prime i.e 5. So the person on the 3rd seat of the 2nd bus is assigned, the $5^3 = 125$ seat of the hotel.

The following bus are powers of 7. The passengers of the n th bus is assign rooms with powers of the n th prime. Since each of these rooms are powers of prime numbers and nothing else, there are no overlapping rooms. There of course room left over with this strategy, but none the less, it does allow the hotel manager to accomodate an infinite number of buses each with an infinite number of passengers.

11. Is the size of the set {Numbers from 0 to 1} equal to the size of the set $\{-\infty$ to $\infty\}$? (Hint: Remember that there are decimal numbers between 0 and 1)

Solution. Every number can be mapped to the a number from $-\infty$ to ∞ by the rule $\tan \frac{\pi}{2}(x - \frac{1}{2})$. There are as many numbers in the interval $(0, 1)$ as there are from $-\infty$ to ∞ .

This solution may come off as out of the blue, but will require an understanding of at least the high school level.

12. Write the Inclusion-Exclusion Principle for $|A \cup B \cup C \cup D \cup E|$.

Solution.

$$\begin{aligned}
 |A \cup B \cup C \cup D| &= |A| + |B| + |C| + |D| \\
 &\quad - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| \\
 &\quad + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| \\
 &\quad - |A \cap B \cap C \cap D|
 \end{aligned}$$