



Grade 7/8 Math Circles

February 28 & March 1, 2017

Fractals

Fractal: A geometric figure that is formed by the repeated application of a certain process (or *iteration*).

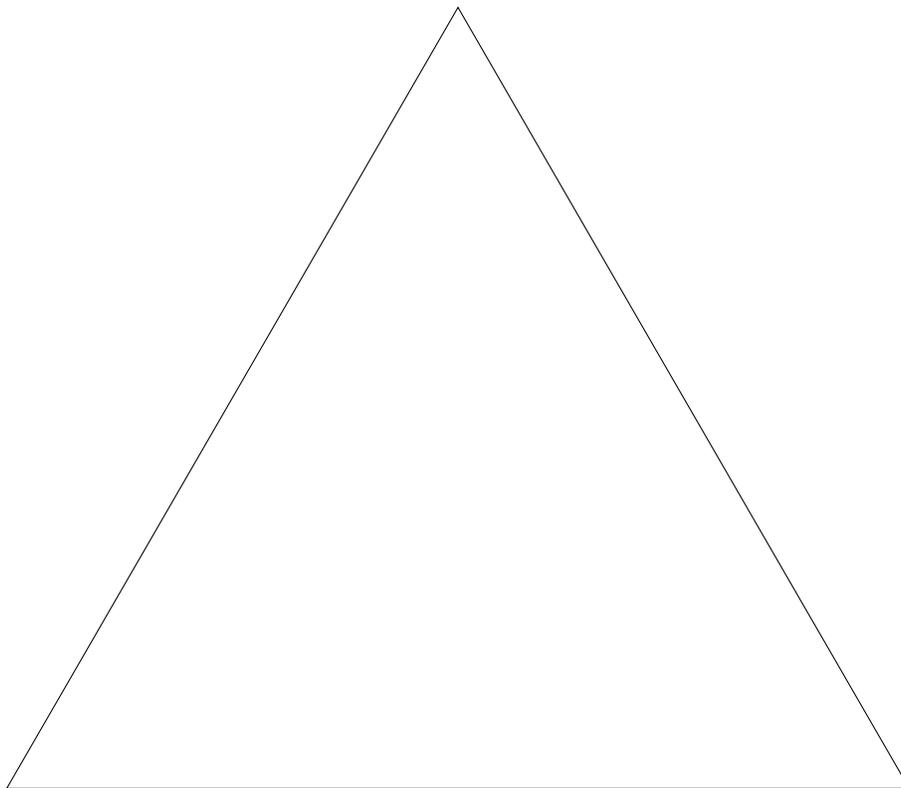
Fractals do not change complexity at any level of magnification. This means its shape looks the same no matter how much we zoom in or out.

To construct a fractal, we use a tool called “*iteration*”. This means applying a step to an original figure (usually a simple geometric shape), taking the result, and applying the step again. With fractals, we repeat this process infinitely many times (it never ends).

Sierpinski Triangle

Today, we will begin by drawing a very well-known fractal called the **Sierpinski Triangle**. Take a few minutes to try drawing the fractal using the following steps. Step 1 has been done for you already.

1. Draw an equilateral triangle
2. Connect the midpoints of all three sides
3. Remove (by shading in or cutting out) the resulting upside-down middle triangle
4. Repeat steps 2-3 with the remaining smaller triangles
5. Repeat step 4 infinitely many times!



Let's look into this shape: What shape do we find if we look at the bottom right corner of Sierpinski's Triangle? We find the Sierpinski Triangle! And if we were to focus on the corner of this corner, we would find the Sierpinski Triangle again!

This property is called *self similarity*, and is present in all fractals.

Self similarity: An object is *self similar* if parts of that object look similar to the whole object

The Sierpinski Triangle is also an example of a *base-motif fractal*.

Base-motif fractal: Each occurrence of a certain shape (the "base") is replaced by a slightly modified version of itself (the "motif")

In the case of Sierpinski, the base is a triangle and the motif is three smaller triangles.

Area of the Sierpinski Triangle

Suppose the area of the initial equilateral triangle is 1 unit².



1. What is the area of the Sierpinski Triangle after 1 iteration?

2. What is the area of the triangle after 2 iterations?

3. What is the area of the triangle after 3 iterations?

Question 1: What is the area of the Sierpinski Triangle after the n -th iteration?

Question 2: What is the area of the Sierpinski Triangle after an infinite number of iterations?

Interesting Property of the Sierpinski Triangle

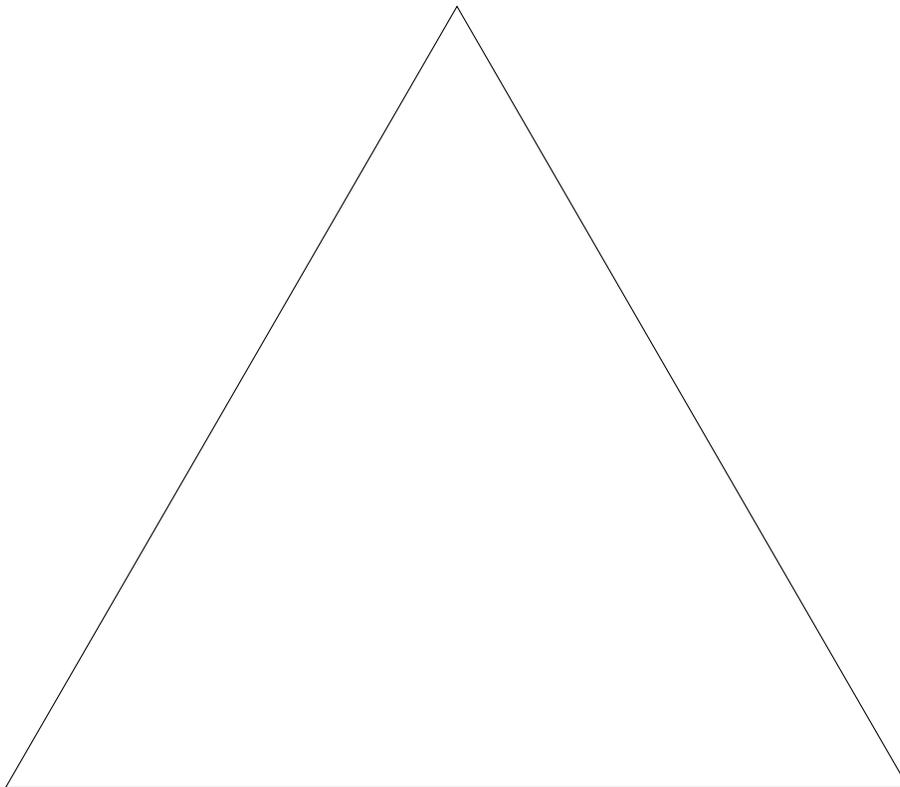
The Sierpinski Triangle has _____ area.

Koch Snowflake

Another famous base-motif fractal is the **Koch Snowflake**.

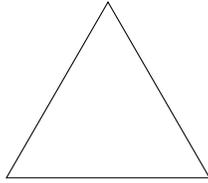
Again, take a few minutes to draw it out!

1. Draw an equilateral triangle
2. Divide each side into three line segments of equal length
3. Using the middle line segment as a base, draw an equilateral triangle (the motif) such that the triangle points outwards
4. Remove the base of the triangle from step 3
5. Repeat step 2-4 with every line segment, both the newly created ones and those left from the previous iteration
6. Repeat step 5 infinitely many times!



Perimeter of the Koch Snowflake

Suppose the side length of the initial equilateral triangle is 1 unit.



1. What is the initial perimeter of the Koch Snowflake?
2. What perimeter of the Koch Snowflake after 1 iteration?
3. What is the perimeter of the Koch Snowflake after 2 iterations?

Question 1: What is the perimeter of the Koch Snowflake after n iterations?

Question 2: What is the perimeter of the Koch Snowflake after infinite iterations?

Area of Koch Snowflake

So we can describe the perimeter of the Koch Snowflake as *infinite*. The perimeter grows larger with each iteration. But what about the area? Without doing any actual calculations, how can we describe the area of the Koch Snowflake?

Consider This! If we draw a circle around the Koch Snowflake, we know the circle has a fixed area. Will the area of the snowflake ever exceed the area of the circle?

Interesting Property of the Koch Snowflake

The Koch Snowflake has a(n) _____ perimeter but has a(n) _____ area.

More Examples of Base-Motif Fractals

Box Fractal

1. Draw a square
2. Divide the square into 9 smaller squares
3. Remove the centre square from each side
4. Repeat steps 2-3 with each of the 5 remaining squares
5. Repeat step 4 infinitely many times!

H-Fractal

1. Draw a horizontal line
2. At both ends of this line, add a perpendicular line that is half as long and intercepts at its midpoint
3. Repeat step 2 for both of the newly added lines
4. Repeat step 3 infinitely many times!

Circle Fractal

1. Draw a circle
2. Inside of your circle, draw two smaller circles that have half the diameter, they should be able to fit snugly side by side in the middle of the larger circle
3. Repeat step 2 with each smaller circle
4. Repeat step 3 an infinite amount of times!

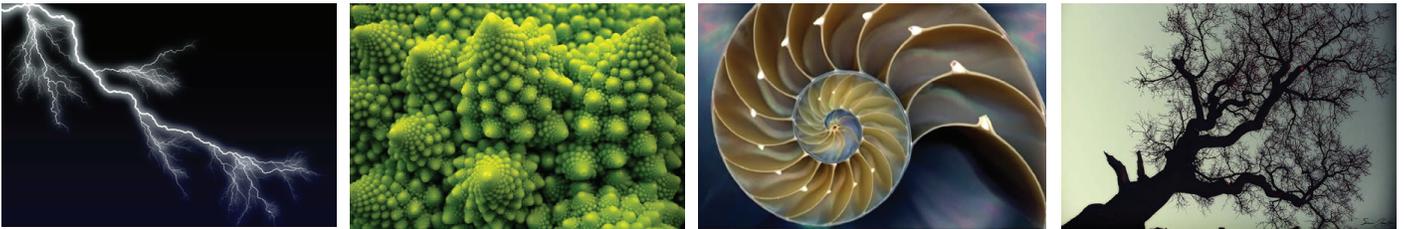
Question: Which of these fractals has an infinite perimeter and an area of 0?

Fractals in the Real World

Fractals are nice to look at but where do we see them in the world?

As it turns out, fractals appear just about everywhere. Here are a few places where you can see fractals...

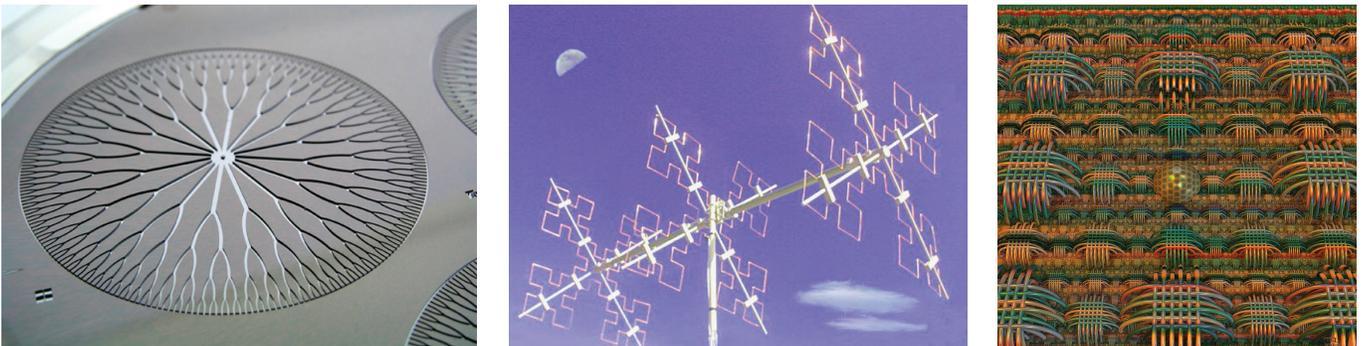
- Art
- Nature (lightning, mountain ranges, shorelines, seashells, trees, ...)



You can see them in artwork and special effects, hear them in fractal music arrangements, and even taste them with foods like broccoflower. Now that you have learned about fractals, it's hard to unsee them now, right?

Okay, so we can see fractals in real life but where do we use them?

- Engineering (ex. cooling circuits, cell phone antennae, fluid mechanics)
- Biology (ex. blood vessels, neurons, bronchial tubes)
- Computer science (ex. image compression, fractal geometry)



Famous Fractals

Cantor Set

Use the following steps to construct a universally famous fractal called the Cantor Set:

1. Draw a horizontal line
2. Directly underneath your line, draw the exact same line, but with the middle piece removed
3. Repeat step 2 with both of your newly created lines
4. Repeat step 3 infinitely many times!

Draw the first 4 iterations.

Why is this fractal universally famous?

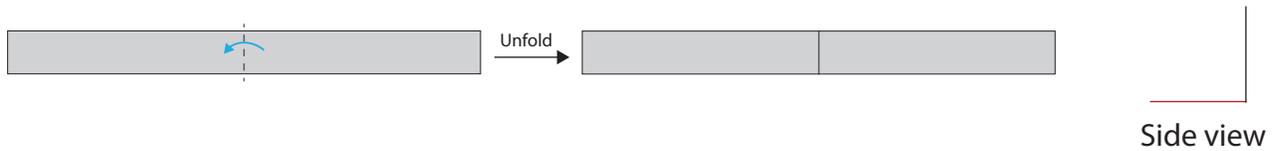
If you draw circles using the points in your last iteration, you will have a simple model of Saturn's ring, which is actually composed of many many self similar rings.

Dragon Curve

This fractal is also known as the **Jurassic Park Fractal** because in a book version of *Jurassic Park*, the each chapter title page has an iteration of this fractal drawn on it!

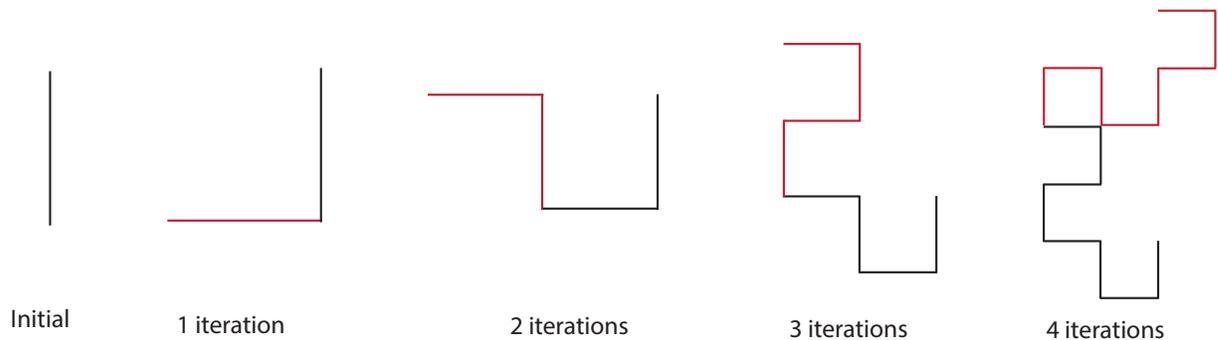
Now, let's create the **Dragon Curve** using a strip of paper and the following steps:

1. Fold your strip of paper in half, then unfold it to make a right angle as shown below. This is your first iteration.

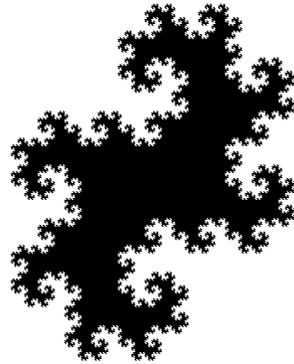


2. Refold your paper, then fold it in half once more *in the same direction* to create a new fold. (That is, if you folded your paper from right to left to start, fold it from right to left again.)
3. Unfold and make sure all corners are right angles. You will see that our new fractal is made up of two copies of our previous iteration stuck together at a right angle
4. Repeat step 2-3 until your paper becomes too difficult to fold (about two or three more iterations). Be sure to unfold and take note of the self similarity after each iteration

The first few iterations are shown below:

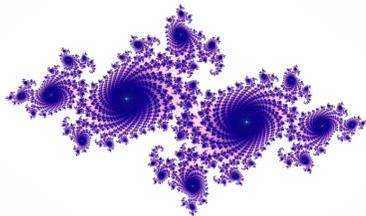
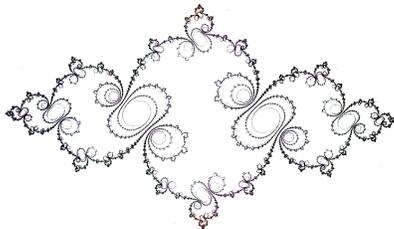


If we were able to fold the paper an infinite amount of times (we can use a computer to simulate this process), the result would be the following figure:

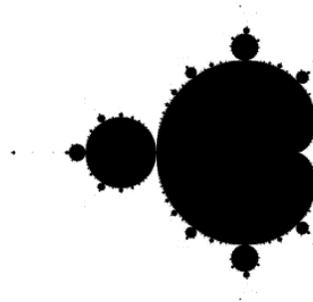


The more difficult we make our iterations, the more beautiful and intricate our figures become. Here are two examples of fractals that use a higher level math concepts:

The Julia Set



The Mandelbrot Set

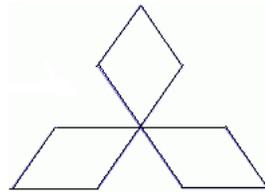


For a really neat video that zooms in on the Mandelbrot set, check out the video below:
<https://www.youtube.com/watch?v=9G6u07ZHtK8>

Problem Set

1. Koch Antisnowflake

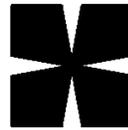
What happens to the Koch Snowflake if we draw our points inside the triangle instead of outside? Here is the first iteration:



Continue for a few more iterations to construct the Koch Antisnowflake.

2. Cesaro

- (i) Draw a filled in square
- (ii) Remove a sharp spike from the centre of each side, resulting in the following motif:



- (iii) Repeat step (ii) with each of the 4 square-like sections of the motif
- (iv) Repeat step (iii) an infinite amount of times!

3. Sierpinski Carpet

- (i) Draw a square
- (ii) Divide the square into 9 smaller squares, fill in the middle square and erase any lines remaining from dividing the square
- (iii) Repeat step (ii) with the 8 remaining squares
- (iv) Repeat step (iii) infinitely many times!

Note: This fractal can be extended to 3 dimensions. Divide a cube into 27 smaller cubes, then remove the centre cube and the cubes at the centre of each side. If you repeat this process with the 20 remaining cubes, you will end up with a 3D fractal called the “Menger Sponge”.

4. Pythagoras Tree

(a) Triangle Roof with $45^\circ - 45^\circ - 90^\circ$

- i. Draw a simple “house” shape (a square with a triangle roof with $45^\circ - 45^\circ - 90^\circ$ on top)
- ii. Add another house shape on top of both sides of your previous house
- iii. Repeat step ii. for the two newly made house shapes
- iv. Repeat step iii. infinitely many times!

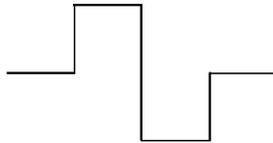
(b) Triangle Roof with $30^\circ - 60^\circ - 90^\circ$

Follow the same steps as (a) but with a triangle roof with $30^\circ - 60^\circ - 90^\circ$ instead. To help you draw this, make sure you...

- ...orient the triangle the same way for every step, and
- alternate which way the triangle will face for each iteration

5. Minkowski Sausage

- (i) Draw a straight line. This is your base
- (ii) Replace your base with the following motif:



- (iii) Repeat step (ii) for all newly created line segments
- (iv) Repeat step (iii) an infinite amount of times!

6. Star of David Fractal

- (i) Draw an equilateral triangle
- (ii) Draw an inverted triangle directly on top of the previous triangle (i.e. make a Star of David)
- (iii) Repeat step (ii) for all six of the smaller triangles you have just created
- (iv) Repeat step (iii) an infinite amount of times!

7. * Suppose the area of the initial equilateral triangle is 4 units².
- (a) What is the area of the Sierpinski Triangle after 2 iterations?
 - (b) What is the area of the Sierpinski Triangle after 5 iterations?
 - (c) If the area of the initial triangle is A_0 units², what is the area of the Sierpinski Triangle after n iterations?
8. * Suppose the side length of the initial equilateral triangle is 2 units.
- (a) What is the perimeter of the Koch Snowflake after 2 iterations?
 - (b) What is the perimeter of the Koch Snowflake after 4 iterations?
 - (c) If the side length of the initial triangle is s units, what is the perimeter of the Koch Snowflake after n iterations?
9. Get creative and make your own fractal!