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Grade 7/8 Math Circles

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Fractals

Fractal: A geometric figure that is formed by the repeated application of a certain process (or *iteration*).

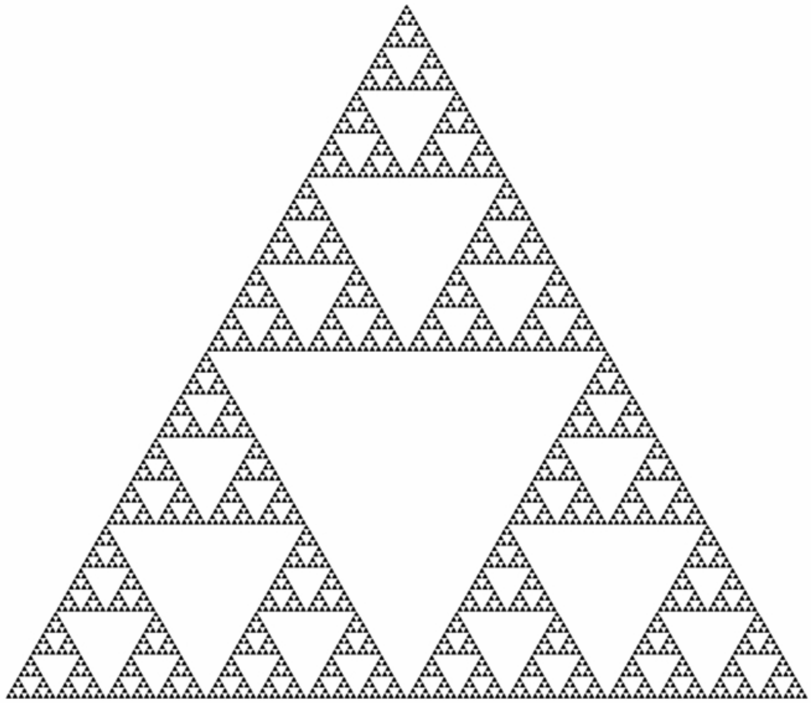
Fractals do not change complexity at any level of magnification. This means its shape looks the same no matter how much we zoom in or out.

To construct a fractal, we use a tool called "*iteration*". This means applying a step to an original figure (usually a simple geometric shape), taking the result, and applying the step again. With fractals, we repeat this process infinitely many times (it never ends).

Sierpinski Triangle

Today, we will begin by drawing a very well-known fractal called the **Sierpinski Triangle**. Take a few minutes to try drawing the fractal using the following steps. Step 1 has been done for you already.

1. Draw an equilateral triangle
2. Connect the midpoints of all three sides
3. Remove (by shading in or cutting out) the resulting upside-down middle triangle
4. Repeat steps 2-3 with the remaining smaller triangles
5. Repeat step 4 infinitely many times!



Let's look into this shape: What shape do we find if we look at the bottom right corner of Sierpinski's Triangle? We find the Sierpinski Triangle! And if we were to focus on the corner of this corner, we would find the Sierpinski Triangle again!

This property is called *self similarity*, and is present in all fractals.

Self similarity: An object is *self similar* if parts of that object look similar to the whole object

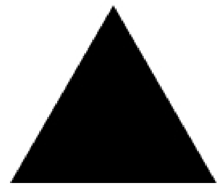
The Sierpinski Triangle is also an example of a *base-motif fractal*.

Base-motif fractal: Each occurrence of a certain shape (the "base") is replaced by a slightly modified version of itself (the "motif")

In the case of Sierpinski, the base is a triangle and the motif is three smaller triangles.

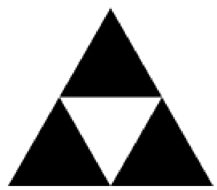
Area of the Sierpinski Triangle

Suppose the area of the initial equilateral triangle is 1 unit^2.



1. What is the area of the Sierpinski Triangle after 1 iteration?

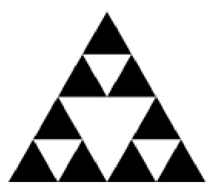
   After 1 iteration, the triangle appears as shown below.

   We removed $\frac{1}{4}$ of the original shape and so the area is now $1 \times \frac{3}{4} = \frac{3}{4}$ units^2.

2. What is the area of the triangle after 2 iterations?

   After 2 iterations, the triangle appears as shown below.

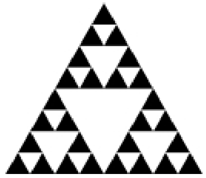
   

   We removed 3 of the 12 smaller triangles after the first iteration, and so are left with $\frac{9}{12} = \frac{3}{4}$ of the first iteration's triangle. This means we now have an area of

$$1 \times \frac{3}{4} \times \frac{3}{4} = \left(\frac{3}{4}\right)^2 = 0.5625 \text{ units}^2.$$

3. What is the area of the triangle after 3 iterations?

   After 3 iterations, the triangle appears as shown below.



   Again, we removed $\frac{1}{4}$ of the smaller triangles so the area of the triangle is now

$$1 \times \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \left(\frac{3}{4}\right)^3 = 0.421875 \text{ units}^2.$$

Question 1: What is the area of the Sierpinski Triangle after the n-th iteration?

If we continue this pattern, we see that at the end of each iteration, we are left with $\frac{3}{4}$ of what we had at the beginning of that iteration. In relation to the original triangle, after the n-th iteration, the area is

$$\mathrm{A}_n = 1 \times \left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^n.$$

Question 2: What is the area of the Sierpinski Triangle after an infinite number of iterations?

Notice that the more iterations performed, the smaller the area becomes. In other words, as n increases to infinity, the area decreases to 0. Thus, we say that the Sierpinski Triangle has an area of 0.

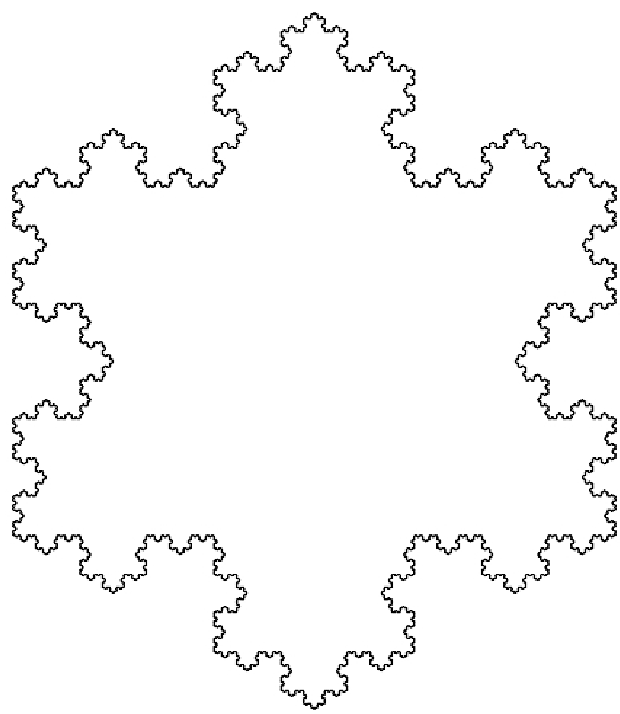
Interesting Property of the Sierpinski Triangle

The Sierpinski Triangle has zero area.

Koch Snowflake

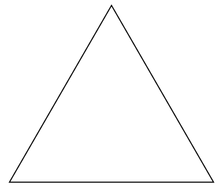
Another famous base-motif fractal is the **Koch Snowflake**.
Again, take a few minutes to draw it out!

1. Draw an equilateral triangle
2. Divide each side into three line segments of equal length
3. Using the middle line segment as a base, draw an equilateral triangle (the motif) such that the triangle points outwards
4. Remove the base of the triangle from step 3
5. Repeat step 2-4 with every line segment, both the newly created ones and those left from the previous iteration
6. Repeat step 5 infinitely many times!



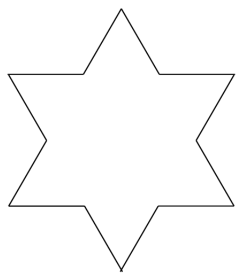
Perimeter of the Koch Snowflake

Suppose the side length of the initial equilateral triangle is 1 unit.



1. What is the initial perimeter of the Koch Snowflake? $P_0 = 1 + 1 + 1 = 3$ units

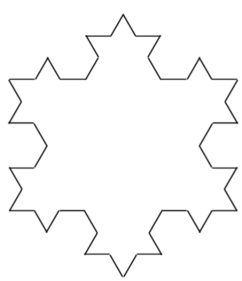
2. What perimeter of the Koch Snowflake after 1 iteration?



Note that each side of the initial triangle is divided into 4 sides of equal length. Now there are $3 \times 4 = 12$ sides after 1 iteration. Since the side of the inital triangle was divided into three equal line segments, the new side length of every side (old and new sides) is $\frac{1}{3}$ the length of the initial side length. Thus, the perimeter is

$$P_1 = 3 \times 4 \times \frac{1}{3} = 12 \times \frac{1}{3} = 4 \text{ units.}$$

3. What is the perimeter of the Koch Snowflake after 2 iterations?



Again, each side is divided into 4 equal sides so now, with respect to the initial triangle, there are $3 \times 4 \times 4 = 3 \times 4^2 = 48$ sides. From the last iteration, each side length is $\frac{1}{3}$

unit. Now again, each new side is $\frac{1}{3}$ the length of the previous side length. So the new side length is $\frac{1}{3} \times \frac{1}{3} = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$. Thus, the perimeter is

$$P_2 = 3 \times 4^2 \times \left(\frac{1}{3}\right)^2 = 48 \times \frac{1}{9} \approx 5.3333 \text{ units.}$$

Question 1: What is the perimeter of the Koch Snowflake after n iterations?

Notice the number of sides of the Koch Snowflake increases by a factor of 4 after each iteration and the side length is divided by 3 after each iteration. We can generalize a formula for the perimeter and say

$$P_n = 3 \times 4^n \times \left(\frac{1}{3}\right)^n = \frac{3 \times 4^n}{3^n}$$

Question 2: What is the perimeter of the Koch Snowflake after infinite iterations?

The perimeter of the Koch Snowflake only increases after each iteration so we can say that the perimeter is infinite. (i.e. $P_\infty = \infty$)

Area of Koch Snowflake

So we can describe the perimeter of the Koch Snowflake as *infinite*. The perimeter grows larger with each iteration. But what about the area? Without doing any actual calculations, how can we describe the area of the Koch Snowflake?

Consider This! If we draw a circle around the Koch Snowflake, we know the circle has a fixed area. Will the area of the snowflake ever exceed the area of the circle?

No! The area of the snowflake is bounded by the circle. It will not grow larger towards infinity like the perimeter. We say the area of the Koch Snowflake is *finite*.

Interesting Property of the Koch Snowflake

The Koch Snowflake has a(n) infinite perimeter but has a(n) finite area.

More Examples of Base-Motif Fractals

Box Fractal

1. Draw a square
2. Divide the square into 9 smaller squares
3. Remove the centre square from each side
4. Repeat steps 2-3 with each of the 5 remaining squares
5. Repeat step 4 infinitely many times!

H-Fractal

1. Draw a horizontal line
2. At both ends of this line, add a perpendicular line that is half as long and intercepts at its midpoint
3. Repeat step 2 for both of the newly added lines
4. Repeat step 3 infinitely many times!

Circle Fractal

1. Draw a circle
2. Inside of your circle, draw two smaller circles that have half the diameter, they should be able to fit snugly side by side in the middle of the larger circle
3. Repeat step 2 with each smaller circle
4. Repeat step 3 an infinite amount of times!

Question: Which of these fractals has an infinite perimeter and an area of 0?

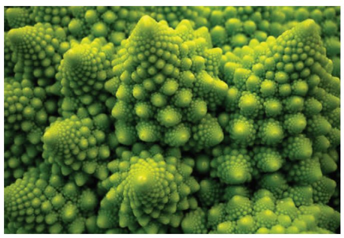
Box Fractal

Fractals in the Real World

Fractals are nice to look at but where do we see them in the world?

As it turns out, fractals appear just about everywhere. Here are a few places where you can see fractals...

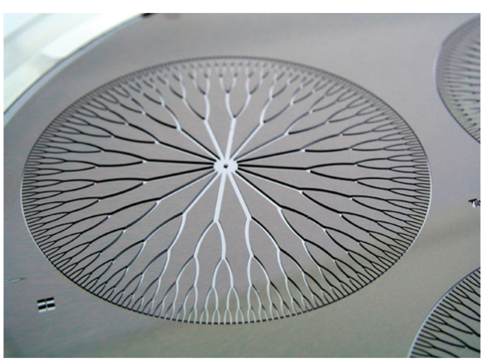
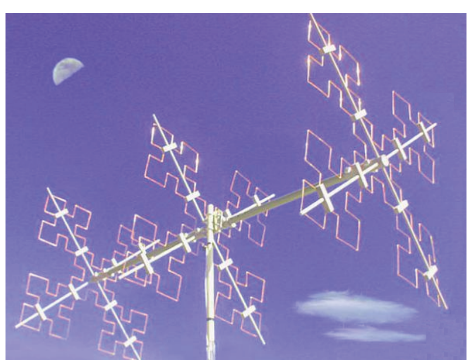
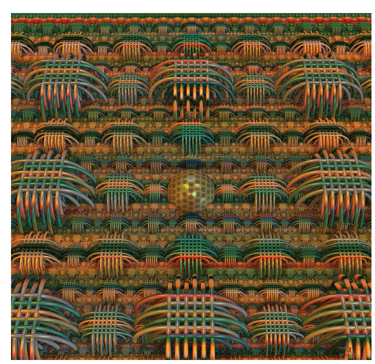
- Art
- Nature (lightning, mountain ranges, shorelines, seashells, trees, ...)

You can see them in artwork and special effects, hear them in fractal music arrangements, and even taste them with foods like broccoflower. Now that you have learned about fractals, it's hard to unsee them now, right?

Okay, so we can see fractals in real life but where do we use them?

- Engineering (ex. cooling circuits, cell phone antennae, fluid mechanics)
- Biology (ex. blood vessels, neurons, bronchial tubes)
- Computer science (ex. image compression, fractal geometry)

Famous Fractals

Cantor Set

Use the following steps to construct a universally famous fractal called the Cantor Set:

1. Draw a horizontal line
2. Directly underneath your line, draw the exact same line, but with the middle piece removed
3. Repeat step 2 with both of your newly created lines
4. Repeat step 3 infinitely many times!

Draw the first 4 iterations.

Why is this fractal universally famous?

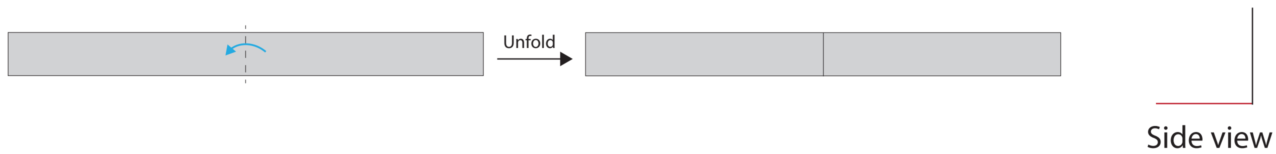
If you draw circles using the points in your last iteration, you will have a simple model of Saturn's ring, which is actually composed of many many self similar rings.

Dragon Curve

This fractal is also known as the **Jurassic Park Fractal** because in a book version of *Jurassic Park*, the each chapter title page has an iteration of this fractal drawn on it!

Now, let's create the **Dragon Curve** using a strip of paper and the following steps:

1. Fold your strip of paper in half, then unfold it to make a right angle as shown below. This is your first iteration.

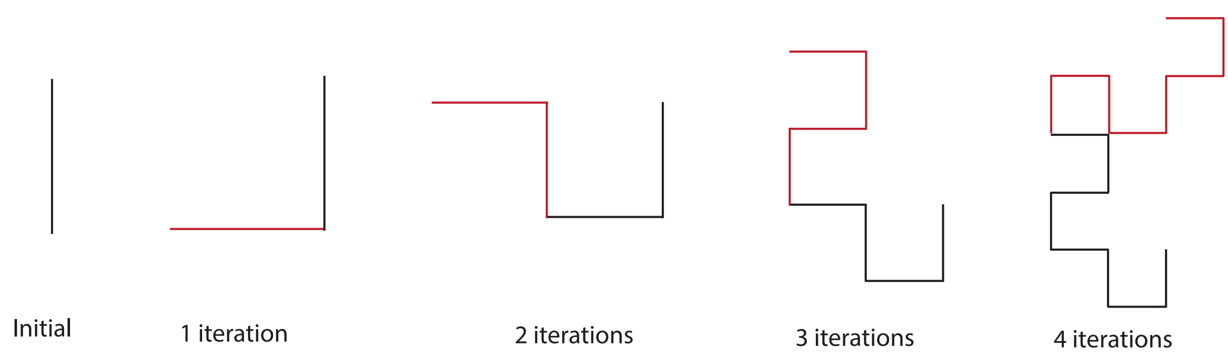


2. Refold your paper, then fold it in half once more *in the same direction* to create a new fold. (That is, if you folded your paper from right to left to start, fold it from right to left again.)

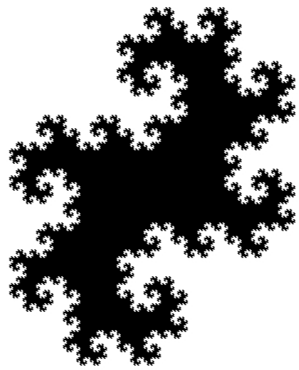
3. Unfold and make sure all corners are right angles. You will see that our new fractal is made up of two copies or our previous iteration stuck together at a right angle

4. Repeat step 2-3 until your paper becomes too difficult to fold (about two or three more iterations). Be sure to unfold and take note of the self similarity after each iteration

The first few iterations are shown below:

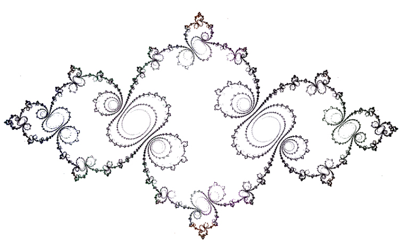


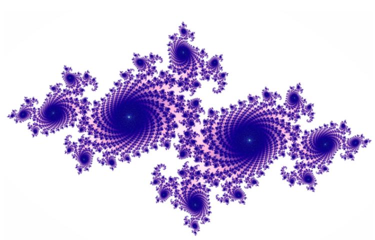
If we were able to fold the paper an infinite amount of times (we can use a computer to simulate this process), the result would be the following figure:



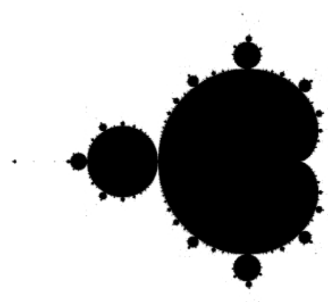
The more difficult we make our iterations, the more beautiful and intricate our figures become. Here are two examples of fractals that use a higher level math concepts:

The Julia Set





The Mandelbrot Set

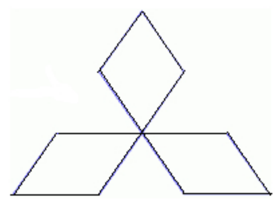


For a really neat video that zooms in on the Mandelbrot set, check out the video below: `https://www.youtube.com/watch?v=9G6uO7ZHtK8`

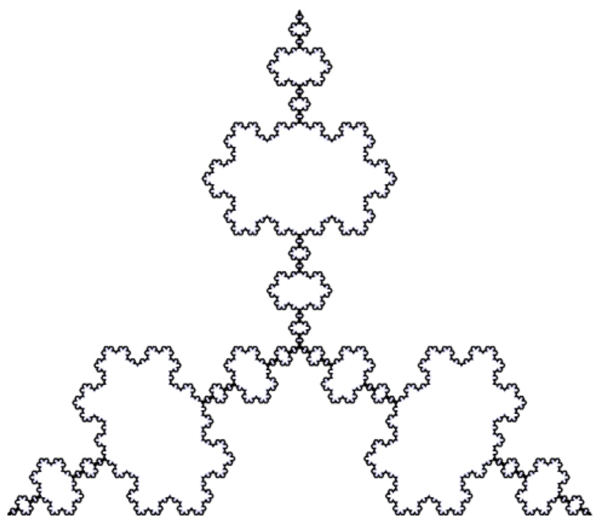
Problem Set Solutions

1. **Koch Antisnowflake**

   What happens to the Koch Snowflake if we draw our points inside the triangle instead of outside? Here is the first iteration:

   Continue for a few more iterations to construct the Koch Antisnowflake.

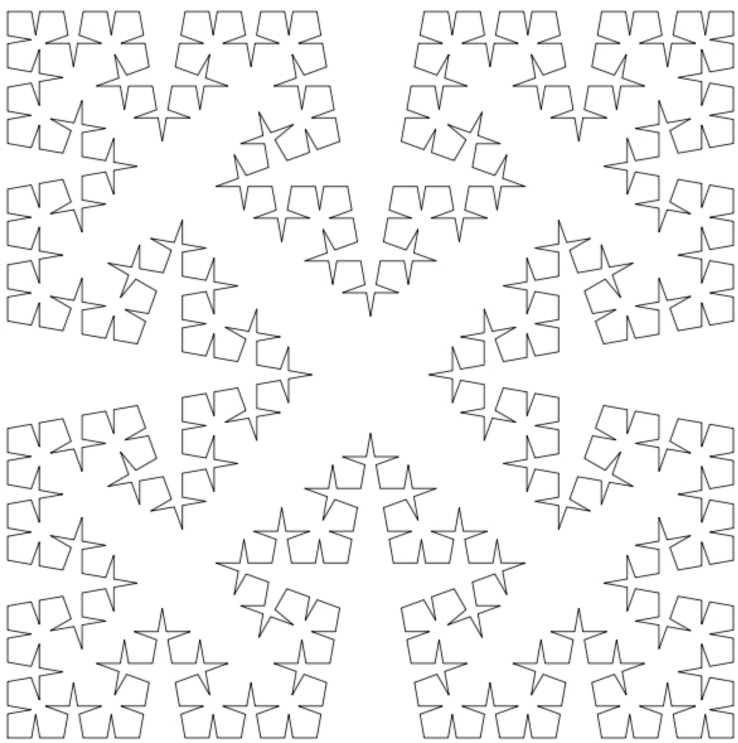
2. **Cesaro**

   (i) Draw a filled in square

   (ii) Remove a sharp spike from the centre of each side, resulting in the following motif:

   (iii) Repeat step (ii) with each of the 4 square-like sections of the motif

   (iv) Repeat step (iii) an infinite amount of times!



3. **Serpinski Carpet**

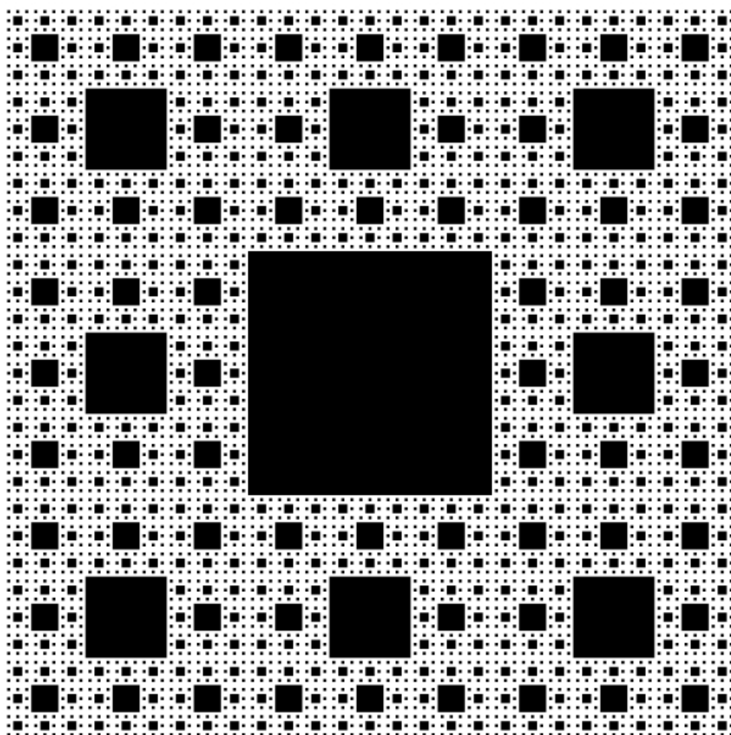
   (i) Draw a square

   (ii) Divide the square into 9 smaller squares, fill in the middle square and erase any lines remaining from dividing the square

   (iii) Repeat step (ii) with the 8 remaining squares

   (iv) Repeat step (iii) infinitely many times!

   Note: This fractal can be extended to 3 dimensions. Divide a cube into 27 smaller cubes, then remove the centre cube and the cubes at the centre of each side. If you repeat this process with the 20 remaining cubes, you will end up with a 3D fractal called the "Menger Sponge".



4. **Pythagoras Tree**

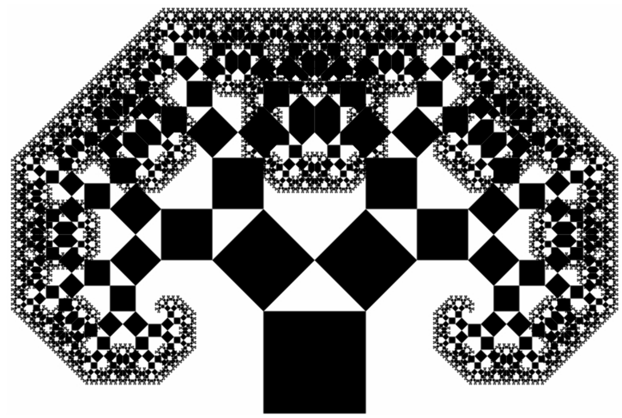
(a) Triangle Roof with 45° - 45° - 90°

i. Draw a simple "house" shape (a square with a triangle roof with 45° - 45° - 90° on top)

ii. Add another house shape on top of both sides of your previous house

iii. Repeat step ii. for the two newly made house shapes

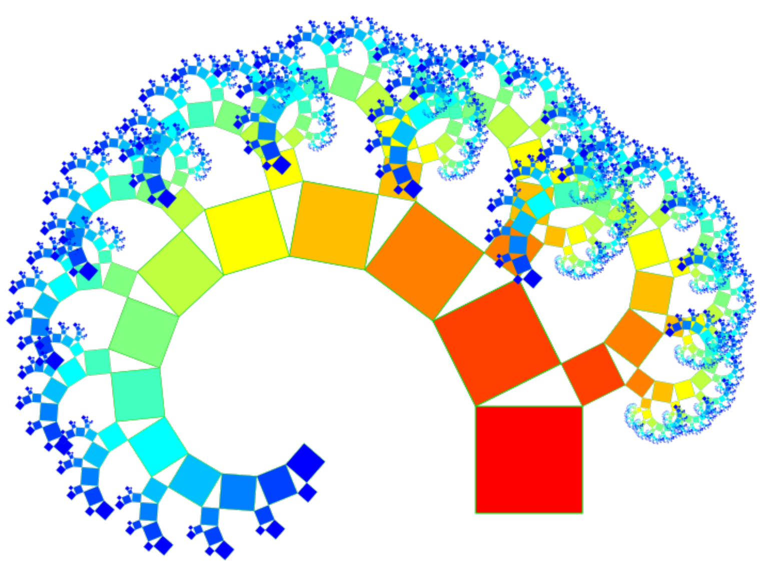
iv. Repeat step iii. infinitely many times!



(b) Triangle Roof with 30° - 60° - 90°

Follow the same steps as (a) but with a triangle roof with 30° - 60° - 90° instead. To help you draw this, make sure you...

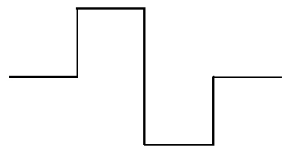
- ...orient the triangle the same way for every step, and
- alternate which way the triangle will face for each iteration



5. **Minkowski Sausage**

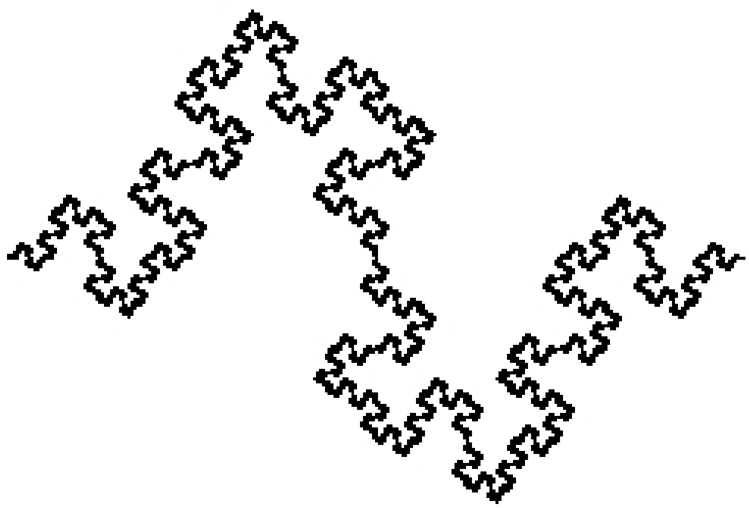
(i) Draw a straight line. This is your base

(ii) Replace your base with the following motif:



(iii) Repeat step (ii) for all newly created line segments

(iv) Repeat step (iii) an infinite amount of times!



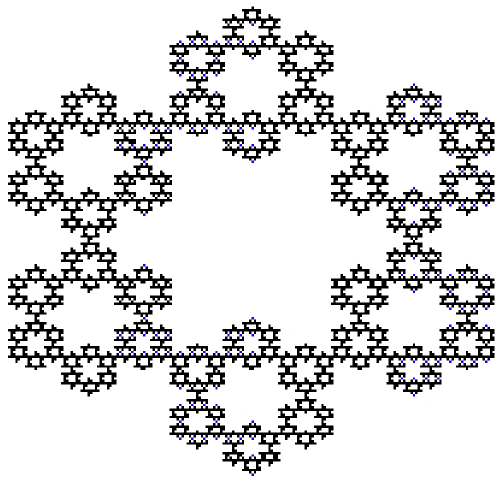
6. **Star of David Fractal**

   (i) Draw an equilateral triangle

   (ii) Draw an inverted triangle directly on top of the previous triangle (i.e. make a Star of David)

   (iii) Repeat step (ii) for all six of the smaller triangles you have just created

   (iv) Repeat step (iii) an infinite amount of times!



7. * Suppose the area of the initial equilateral triangle is 4 units^2.

   (a) What is the area of the Sierpinski Triangle after 2 iterations?

$$A_2 = 4 \times \left(\frac{3}{4}\right)^2 = 4 \times \frac{9}{16} = \frac{9}{4} = 2.25 \text{ units}^2$$

   (b) What is the area of the Sierpinski Triangle after 5 iterations?

$$A_5 = 4 \times \left(\frac{3}{4}\right)^5 = 4 \times \frac{243}{1024} \approx 0.9492 \text{ units}^2$$

   (c) If the area of the initial triangle is A_0 units^2, what is the area of the Sierpinski Triangle after n iterations?

$$A_n = A_0 \times \left(\frac{3}{4}\right)^n$$

8. * Suppose the side length of the initial equilateral triangle is 2 units.

   (a) What is the perimeter of the Koch Snowflake after 2 iterations?

   $$P_2 = 2 \times \frac{3 \times 4^2}{3^2} = 2 \times \frac{48}{9} \approx 10.667 \text{ units}$$

   (b) What is the perimeter of the Koch Snowflake after 4 iterations?

   $$P_5 = 2 \times \frac{3 \times 4^4}{3^4} = 2 \times \frac{768}{81} \approx 18.962 \text{ units}$$

   (c) If the side length of the initial triangle is s units, what is the perimeter of the Koch Snowflake after n iterations?

   $$P_n = s \times \frac{3 \times 4^n}{3^n} = \frac{3 \times 4^n}{3^n} s$$

9. Get creative and make your own fractal!