UNIVERSITY OF
WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

March 21st/22nd, 2017

Geometric Arithmetic

Ancient Greece has given birth to some of the most important features of Western civilization, including democracy and the modern deductive method. But for all of their amazing achievements, they lacked one really important feature - an effective number system. In fact, it is even more amazing that they did all that they did WITHOUT numbers.

The Greeks understood how numbers interacted... but they didn't have a way to write them using symbols. Instead, they had to write them out in *long hand*.

For example, the Greeks would write the algebraic problem

" Solve $2x + 5 = 13$ "

as

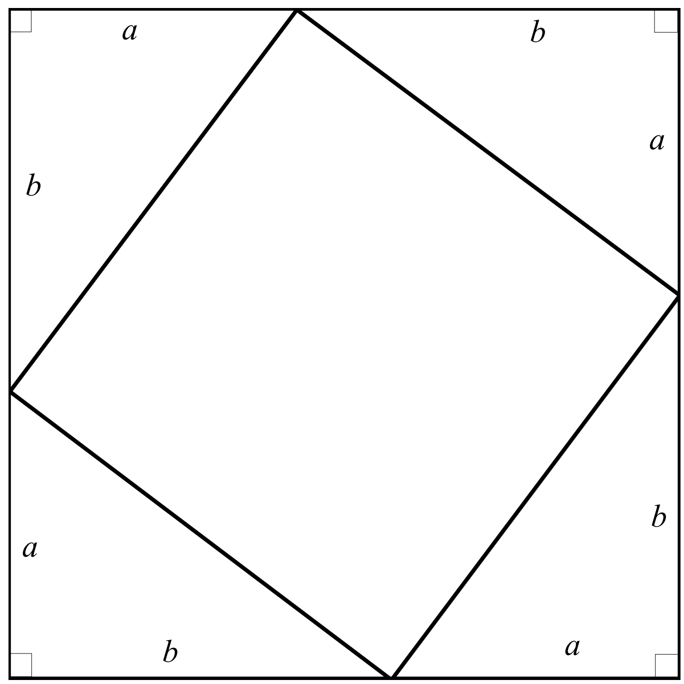
> *Let five added to twice an unknown quantity be equal to thirteen. What is the unknown quantity?*

Not very pleasant to read, right? Imagine having to do this for all of your math problems!

Why was this the case? There are many reasons. One reason is that the Greeks often viewed numbers not as representations of an **amount**, or "how many/much", but rather, as measuring **length**. This was evident in their approach to geometry - geometry was one of their ways of representing numbers! The Greeks did **arithmetic with geometry** - which is what we will be learning today!

The Pythagorean Theorem

One proof of the Pythagorean Theorem involves the following configuration:



To prove the theorem, we need one important fact.

For any two triangles, if two **corresponding** sides are the same **length**, and the **angle** between the two sides are the same in either triangle, then...

1. Both triangles have the same corresponding ANGLES.
2. Both triangles have the same corresponding SIDE LENGTHS
3. In other words, they are the same triangle.

Example.

In order to prove the Pythagorean Theorem, we first need to show that the four sided shape in the center is actually a square.

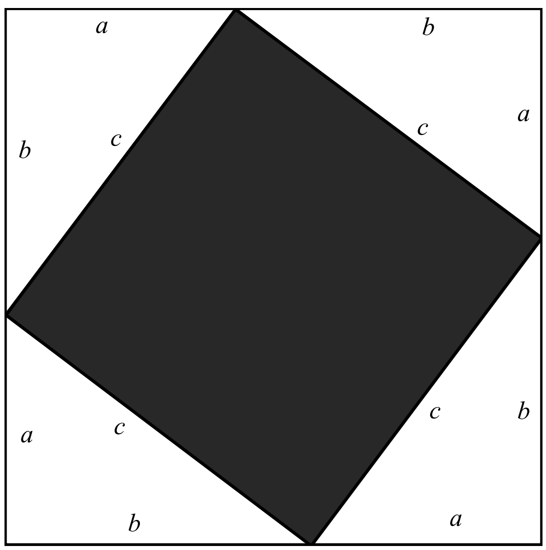
This requires two steps:

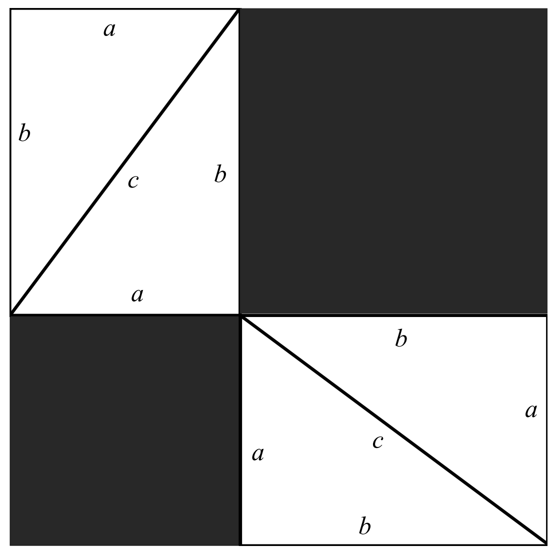
1. We need to show the side lengths are all equal.

2. We need to show that the angles in the quadrilateral are all equal.

The Proof

1. In the left diagram, what is the area of the dark square in terms of c? c^2
2. Imagine moving the triangles around until you get the diagram on the right:





3. What is the total area of the dark region?

   Solution. c^2

4. What is the area of the smaller dark square?

   Solution. a^2

5. What is the area of the larger dark square?

   Solution. b^2

6. What can you conclude? Write it out algebraically.

   Solution. $a^2 + b^2 = c^2$

Something even more interesting...

Notice that...

Solution.

1. $1 = 1 = 1^2$
2. $1 + 3 = 4 = 2^2$
3. $1 + 3 + 5 = 9 = 3^2$
4. $1 + 3 + 5 + 7 = 16 = 4^2$
5. $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
6. $1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$
7. $1 + 3 + 5 + 7 + 9 + 11 + 13 = 49 = 7^2$

$$\vdots$$

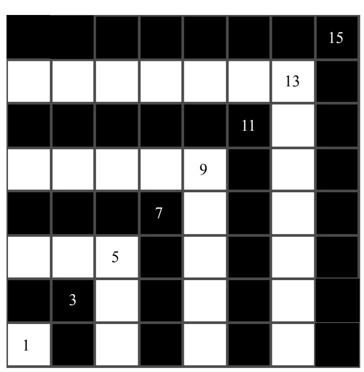
n. $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

Pattern: **Solution.** The SUM of consecutive ODD numbers is a perfect SQUARE.

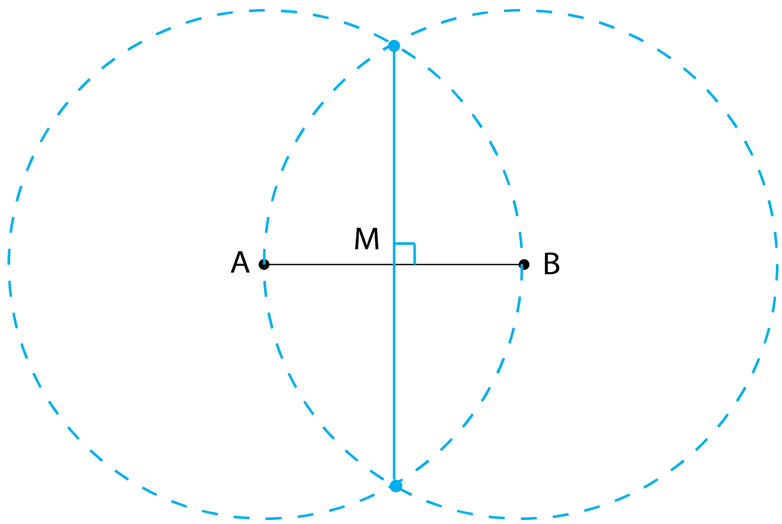
Geometric Proof

Proof by picture.

Let 1 represent a square of area 1, 3 represent 3 squares of area 1, n represent n squares of area 1... etc.

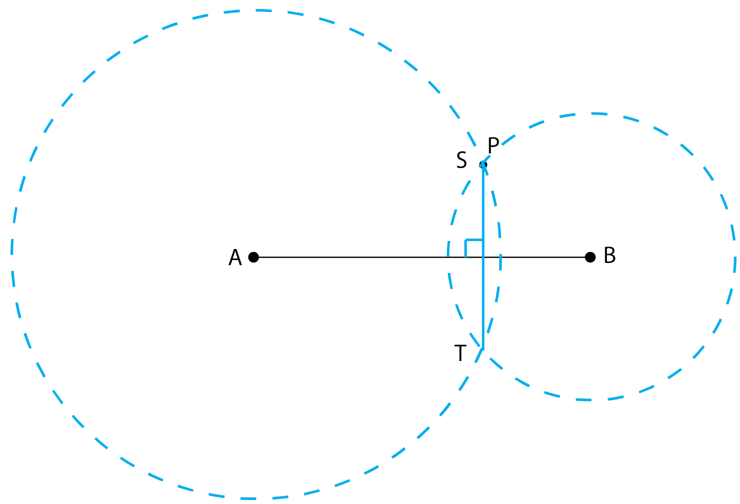


Solution.



1. Widen your compass so that one end is at A and the pencil end on point B. Draw a circle.
2. Repeat step 1, except put one end on the point on B, the pencil end on a point A. Draw a circle
3. These two circles should intersect in two points on the page. Mark these points, then use the straightedge to connect them. Erase the circles.
4. The line you just made should have crossed your original line. This is the perpendicular bisector.
5. Mark the point that your perpendicular bisector cuts your original line as the point M.
6. M is midway between A and B - it is the midpoint!

Solution.



1. Set your compass to have center at A, pencil at P. Draw the circle.

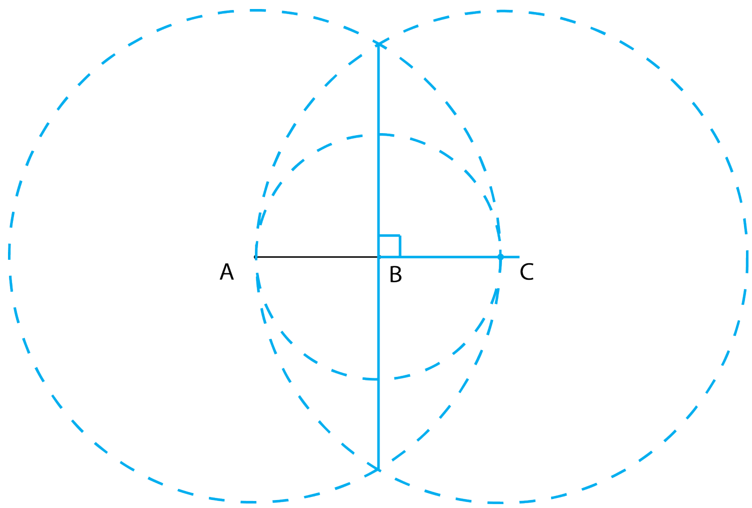
2. Set your compass to have center at B, pencil at P. Draw the circle.

3. There should be two points of intersection of the two circles you drew. Mark them as S and T, then draw a line between them. Erase the circles.

4. This is the perpendicular line that passes through P!

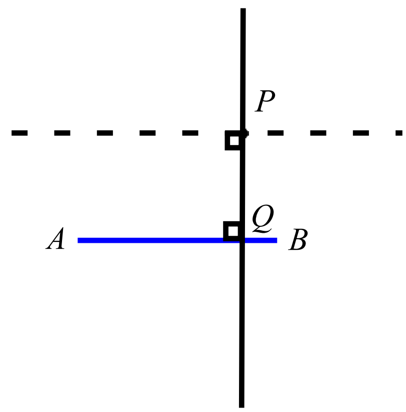
Constructing Perpendicular Through an Endpoint

Solution.



1. Extend the line AB with your straightedge, at least double the length.
2. Use your compass to draw a circle with center at B and radius AB (so put your pencil at A).
3. Mark the point of intersection of that circle with the line. Call that point C. Erase the circles. This forms a line segment, AC.
4. Place the center of your compass at C, the pencil at A. Draw the circle. Repeat, with the center at A and pencil at C. Connect the two points like you did in the midpoint construction. Erase all the circles but KEEP the perpendicular line.
5. The line that you made will pass through B, and it is the perpendicular through B.

Solution.



1. Using the necessary skill "Constructing Perpendiculars to an Outside Point", construct a perpendicular line to AB that passes through P. Erase everything but the perpendicular line.

2. Mark the point of intersection of your perpendicular line and AB as a point Q.

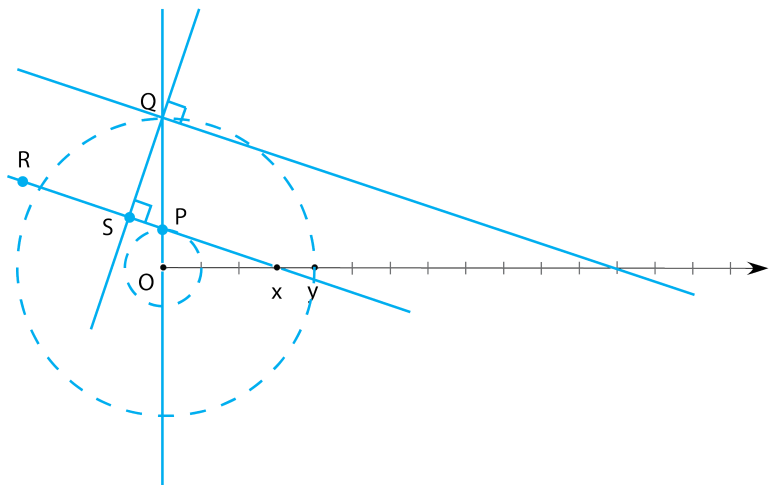
3. Rotate your page, if you wish. Using the line segment PQ, construct a perpendicular line to PQ that passes through P (the endpoint) like you learned before. Erase all unnecessary marks when done.

4. That line is parallel to AB!

Math without Numbers!

Multiplication

Solution.



1. Using what you learned to construct perpendiculars through endpoints, create a perpendicular through the point marked O - make it tall! Erase everything, EXCEPT this perpendicular line.

2. Draw a circle of radius 1 unit (use the dashes) with center at O. Mark the TOPMOST point of intersection of this line with the circle you just drew as point P. Erase the circle.

3. Draw a line using straightedge through P and x.

4. Draw a circle of radius y with center at O. Mark the TOPMOST point of intersection of this circle and the perpendicular through O as point Q. Erase the circle.

5. Refer back to your line Px. Extend this line upwards and to the left; mark a point about 4 cm to the left of P, call it R.

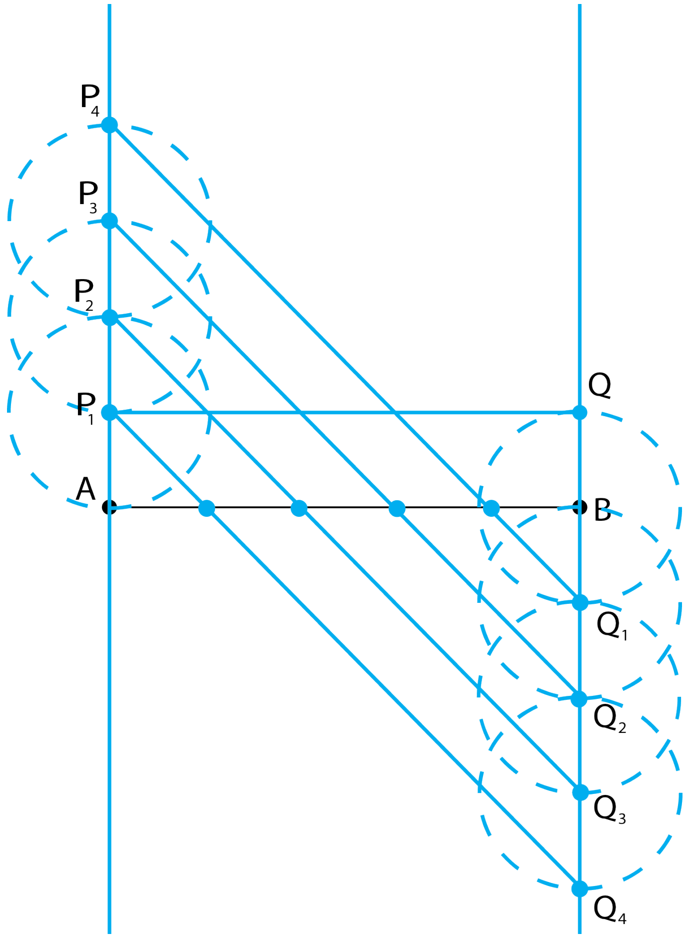
6. Using the line segment RP and the point Q, construct a perpendicular line to RP that passes through Q (you've learned how to do this!). Mark the point of intersection of this line with RP as the point S. (This is getting messy now - you should probably erase all your circles!).

7. Using SQ as your line segment, draw a perpendicular line that passes through Q (refer back to the necessary skill - drawing a perpendicular through an endpoint).

8. Extend this line until it hits the line with the arrow. Mark the point of intersection of the line you just drew with the horizontal line as point xy.

9. Measure the lengths of x, y, and xy using the dashes. How long is xy? What is the relationship?

Solution.



At every step, erase all unnecessary markings to keep it neat!

1. Using what you learned in your necessary skills (perpendicular through an endpoint), draw a perpendicular through A and another one through B.

2. On the top half of the perpendicular through A, pick a point roughly 2 cm (I'm serious, do NOT pick more than 5 cm) above A and mark it as P_1.

3. Put the center of your compass at P_1, pencil at A, and draw a circle. Mark the **other** point of intersection with the perpendicular as P_2.

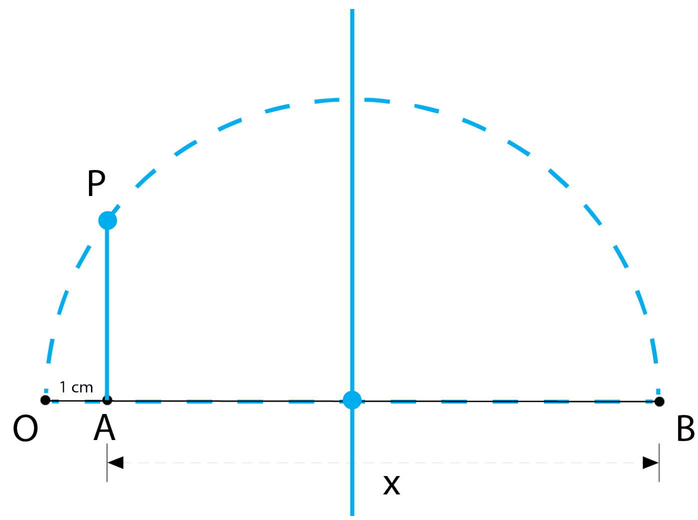
4. Put the center of your compass at P_2, pencil at P_1, and draw a circle. Mark the **other** point of intersection with the perpendicular as P_3.

5. Repeat steps 3 and 4 until you have made point P_4. These points should be equally spaced.

6. Use the necessary skills you've learned to draw a line parallel to AB that passes through P_1 (if you know how to cheat, go ahead here, it's not that important for this step).
7. Mark the point of intersection of this line with the perpendicular through B as a point Q.
8. Draw the circle with radius BQ, with center at B. Mark the **other** point of intersection of this circle with the perpendicular through B as Q_1.
9. Repeat steps 2 - 5, except using Q_1, Q_2, etc. until you've made Q_4.
10. Join P_1 to Q_4, P_2 to Q_3, P_3 to Q_2, and P_4 to Q_1, with a straightedge. Mark the points where these lines intersect the line AB.
11. Those points you marked above divide AB into segments. Measure the length of the entire segment, and then measure the lengths of each small segment - what do you notice? What arithmetic operation have you performed?

Square Root ($x \to \sqrt{x}$)

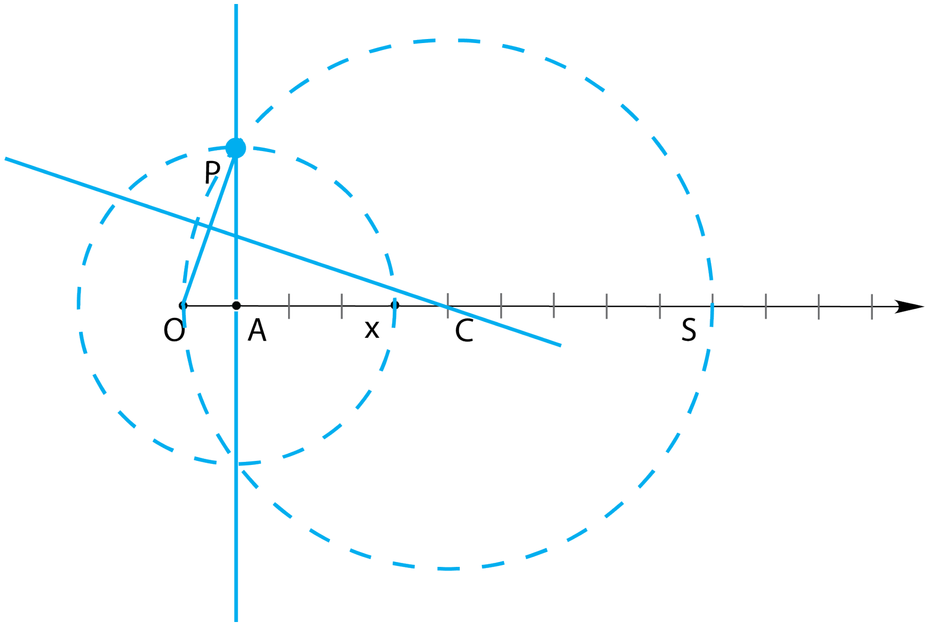
Solution.



1. Construct a point A 1 centimeter to the right of O.
2. Construct the midpoint of the line OB. Mark this point as D. Erase everything else.
3. Construct a semi-circle of radius OD with center at D.
4. Draw a perpendicular through A. Mark the point of intersection of the perpendicular with the semi-circle as the point P.
5. Measure the length of AB, label this x.
6. Measure the length of AP.
7. The length of AP is the square root of x!

Squaring ($x \to x^2$)

Solution.



1. Use the segment OA as your line segment. Draw a perpendicular to OA passing through its endpoint, A.

2. Place the center of your compass at A, pencil at x, and draw a circle. Mark the point of intersection of this circle with the perpendicular you just constructed, as the point P. Erase the circle.

3. Draw a line from O to P.

4. Find the midpoint of the line segment OP. Do not erase the perpendicular bisector.

5. Using the perpendicular bisector your just made in the previous step, extend it until it crosses the arrowed line. Label the point of intersection as C.

6. Construct a circle of radius CO with center at C. Label the point of intersection of this circle with the arrowed line as point S.

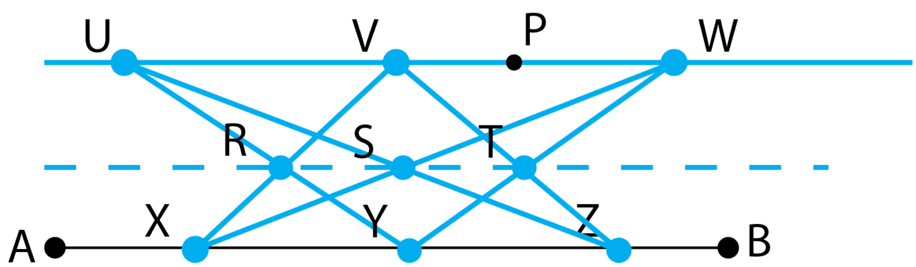
7. Measure the length of Ax using the dashes (distance between two dashes is 1 unit). Measure the length of AS.

8. What is the relationship between the lengths you just measured? You have just squared the length of Ax!

Some Cooler Constructions...

Pappus' Theorem

Solution.

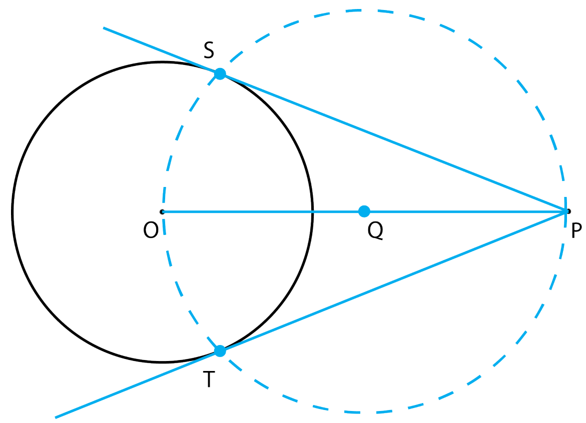


Instructions

1. Construct the parallel line to AB that passes through P (refer back to necessary skills)
2. Mark three random points on the line AB (between A and B!). Call these X, Y, and Z, from left to right.
3. Mark three random points on the line through P. Call them U, V, W, from left to right.
4. Construct the following lines: UY, UZ; VX, VZ; WX, WY.
   (a) Label the intersection of UY and VX as point R.
   (b) Label the intersection of UZ and XW as point S.
   (c) Label the intersection of WY and VZ as point T.
5. Connect points R, S, and T. (Isn't it amazing that they lie on a straight line?)

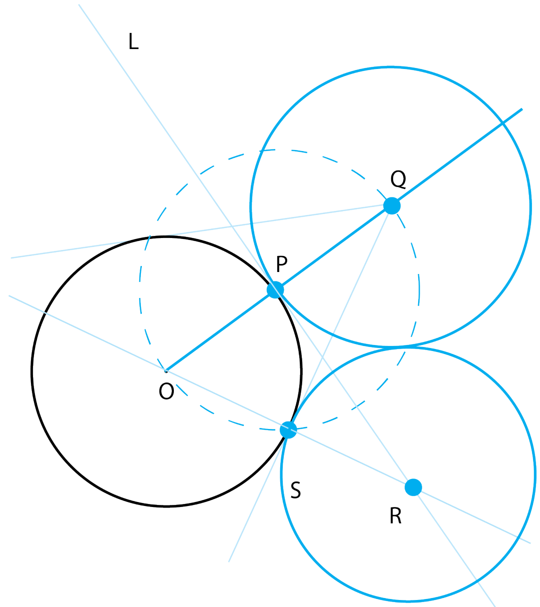
Try with different points! You will ALWAYS get the same result.

Solution.



1. Connect O to P with a straightedge, forming segment OP.
2. Find the midpoint of the segment OP, call it Q.
3. Use your compass to construct the circle of radius QP with center at Q.
4. This circle will intersect your original circle in two points. Mark these points as S and T.
5. Draw a line from P to S, and a line from P to T. These are the tangent lines to the circle that pass through P.

Solution.



1. Pick any point P on the circle's circumference.

2. Draw the line OP, make it fairly long.

3. Draw a circle with radius OP, with center at P. Mark the point of intersection of the circle with the line you made in Step 2 as Q.

4. Draw the circle of radius QP with center at Q. This circle is tangent to the first!

5. Draw a line perpendicular to OQ that passes through P. Call this line L. Extend this line in both directions.

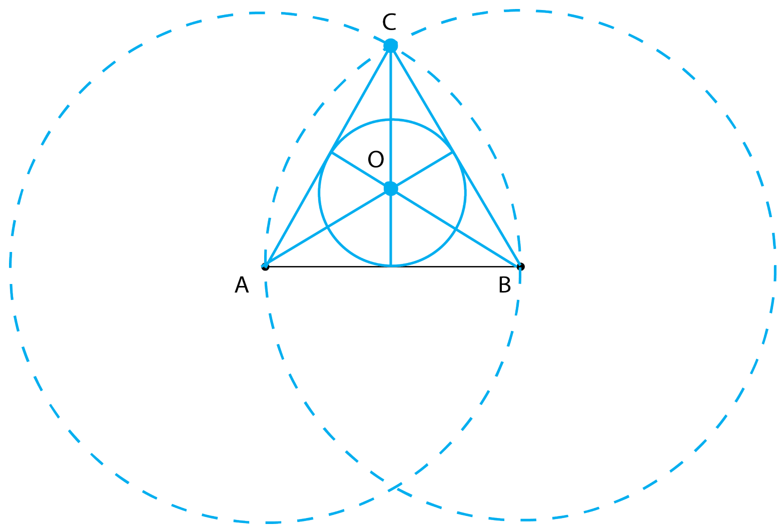
6. Construct the two tangents from Q to the first circle (you learned this in the previous step). Pick one of the points of tangency and call it S.

7. Draw the line OS until it passes through L. Mark this point as R.

8. Draw the circle with radius RS with center at R. This circle will be tangent to the other two circles!

Circle within a Triangle... within a Circle!

Solution.



1. Draw a circle with radius AB with center at A.
2. Draw a circle with radius BA with center at B.
3. Mark the topmost point of intersection as C. Draw the triangle formed by these three points. This is an equilateral triangle!
4. Erase everything in your drawing except for the equilateral triangle.
5. Find the midpoints of each of the sides of the triangle (3 total. (Refer back to necessary skills)
6. From each point, draw a line to the vertex of the triangle that is directly opposite of it. You should have three lines that meet at one point. Call this point O.
7. Pick one of the midpoints. Put the center of your compass at O, the pencil at the midpoint. Rotate your compass.
8. You have just made a circle inside a triangle which is inside two circles!

Challenge Investigation: $(a+b)^2 = ?$

An important algebra fact that you will learn in high school is that if you have two whole numbers a and b, then $(a+b)^2 = a^2 + 2ab + b^2$.

(Algebraic) Proof:

$$(a+b)^2 = (a+b)(a+b)$$

Let $x = (a+b)$. Then

$$\begin{aligned}
(a+b)^2 &= (a+b)(a+b) \\
&= x(a+b) \\
&= x \times a + x \times b && \text{(Distributive Property)} \\
&= (a+b)(a) + (a+b)(b) && \text{(Substitute } x = (a+b)\text{)} \\
&= [a \times a + b \times a] + [a \times b + b \times b] && \text{(Distributive Property again)} \\
&= a^2 + ba + ab + b^2 \\
&= a^2 + 2ab + b^2 && (ab = ba)
\end{aligned}$$

(Fill in the remaining steps of the algebraic proof.)

Can you prove the same result by using geometrical arguments and the following diagram?