UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

March 28 & 29, 2017

Math Jeopardy II

Number Theory

\$100 Is 37 an emirp?

Recall that the definition of an emirp is a prime number that results in a different prime number when it's digits are reversed. Reversing 37 and we have 73 which is also prime.

$$\therefore 37 \text{ is an emirp.}$$

\$200 What is 10000001^3? 1000000300000030000001

\$300 Find the prime factorization of 1470.

There are a variety of strategies that can be used to find the prime factorization.

$$1470 = 2 \times 3 \times 5 \times 7^2$$

\$400 How many factors does 245 have?

To determine how many factors 245 has, we need to first find the **prime factorization**.

$$245 = 5 \times 7^2$$

Then all the possible factors are $2 \times 3 = 6$

Alternatively, it is possible to exhaust all factors of 245. They are as follows: 1,5, 7, 35, 49, 245.

\$500 How many integers n are there when $1 \leq n \leq 100$ and n^n is a perfect square.

There are a total of 55 possible integers. We shall break it into two cases.

Even Case: Notice that an even number to the exponent of the same even number is always a perfect square. For example, using our exponent laws, we can show that 4^4 is a perfect square.

$$4^4 = 4^{2\times 2} = (4^2)^2$$

In general, to show that an even number to the exponent of an even number is a perfect square, we note that every even integer, n, can be expressed as $n = 2m$, where m is another integer.

$$(2m)^{2m} = (2m)^{2\times m} = ((2m)^m)^2$$

We have shown that an even integer to an exponent of an even integer is indeed a perfect squares, and for values between 1 and 100, there are 50 even integers.

Odd Case: There are actually 5 odd integers that when taken to the exponent to itself is an perfect square. In fact they are all the odd perfect squares: 1, 9, 25, 49, 81.

For example, we can show that 9^9 is a perfect square using our exponent laws.

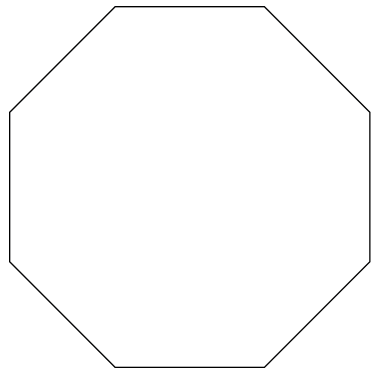
$$9^9 = (3^2)^9 = (3^9)^2$$

. Hence there are 5 odd possible integers.

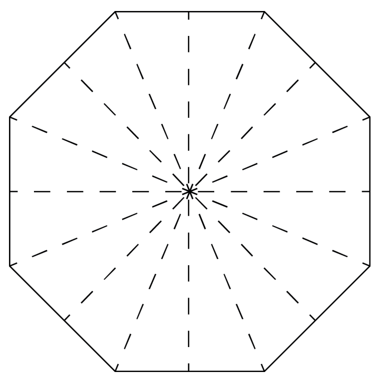
$\therefore$ in total, we have 55 possible integers between 1 and 100.

Spatial Visualization and Origami

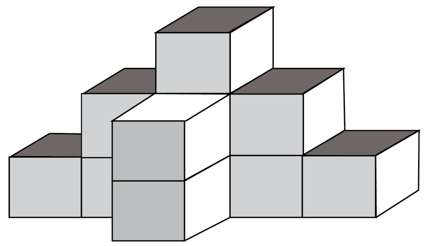
$100 How many lines of symmetry does an octagon have?



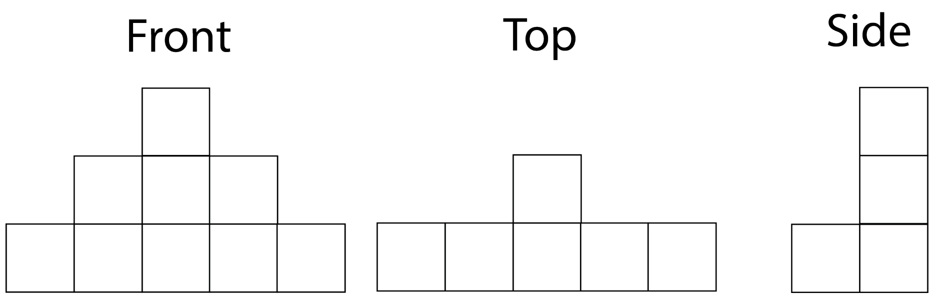
There are 8 lines of symmetry. They are drawn below. In fact for any n-regular polygon, there are n lines of symmetry.



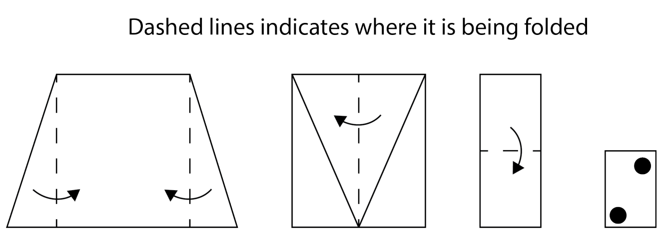
\$200 A 3D shape constructed by blocks is shown below. What are the side, top, and front views of the shape?

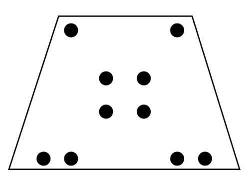


There are 16 lines of symmetry. They are drawn below:

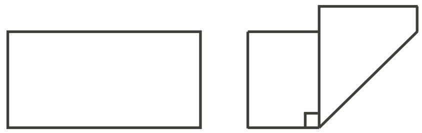


\$300 Suppose we fold a figure in the sequence shown below. Suppose we punch two black holes in the last step. What does the figure look like after we unfold it?





\$400 A rectangular piece of paper measures 17 cm by 8 cm. It is folded so that a right angle is formed between the two segments of the original bottom edge, as shown. What is the area of the new figure?



When the paper is folded in this way, the portion of the original bottom face of the paper that is visible has the same area as the original portion of the top side of the paper to the right of the fold. (This is quadrilateral $CDEF$.)

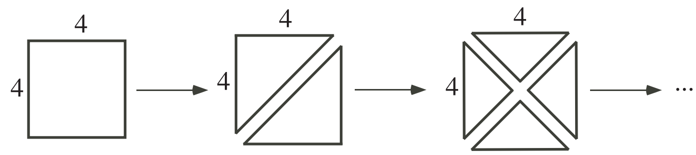
Of the portion of the original sheet to the left of the fold, the part that is hidden (and thus not included in the area of the new figure) is the triangular portion under the folded part. (This is the section under $\triangle CDG$.) The hidden triangle is congruent to $\triangle CDG$.

Thus, the area of the portion of the original top face of the paper that is visible is the area to the left of the fold, minus the area of the hidden triangle. Therefore, the area of the new figure equals the area of the original rectangle minus the area of $\triangle CDG$.

$\triangle CDG$ has height $GC = 8$ cm (the height of the rectangle), is right-angled (since the folded portion of the original bottom edge is perpendicular to the top and bottom edges), and has base $GD = 8$ cm. Therefore, $\triangle CDG$ has area $\frac{1}{2}(8)(8) = 32\text{ cm}^2$.

$\therefore$ the area of the new figure is $136 - 32 = 104\text{ cm}^2$

\$500 A 4×4 square piece of paper is cut into two identical pieces along its diagonal. The resulting triangular pieces of paper are each cut into two identical pieces.



Each of the four resulting pieces is cut into two identical pieces. Each of the eight new resulting pieces is finally cut into two identical pieces. What is the length of the longest edge of the final sixteen pieces of paper?

We label the stages in this process as Stage 0 (a square), Stage 1 (2 triangles), Stage 2 (4 triangles), Stage 3 (8 triangles), and Stage 4 (16 triangles).

We want to determine the length of the longest edge of one of the 16 triangles in Stage 4. At Stage 1, we have two right-angled isosceles triangles with legs of length 4. Consider a general right-angled isosceles triangle ABC with legs AB and BC of length a. Since this is a $45° - 45° - 90°$ triangle, its hypotenuse AC has length $\sqrt{2}a$. We split the triangle into two equal pieces by bisecting the right-angle at B.

Since $\triangle ABC$ is isosceles, then this bisecting line is both an altitude and a median. In other words, it is perpendicular to AC at M and M is the midpoint of AC. Therefore, the two triangular pieces $\triangle AMB$ and $\triangle CMB$ are identical $45° - 45° - 90°$ triangles. The longest edges of these triangles (AB and CB) are the legs of the original triangle, and so have length a.

Since the longest edge of the original triangle was $\sqrt{2}a$, then the longest edge has been reduced by a factor of $\sqrt{2}$ Since we have shown that this is the case for an arbitrary isosceles right-angled triangle, we can then apply this property to our problem. In Stage 1, the longest edge has length $4\sqrt{2}$.

Since the longest edge in Stage 1 has length $4\sqrt{2}$, then the longest edge in Stage 2 has length $\frac{4\sqrt{2}}{\sqrt{2}} = 4$. Since the longest edge in Stage 2 has length 4, then the longest edge in Stage 3 has length $\frac{4}{\sqrt{2}} = \frac{2\sqrt{2}\sqrt{2}}{\sqrt{2}} = 2\sqrt{2}$.

Since the longest edge in Stage 3, has length $2\sqrt{2}$, then the longest edge in Stage 4 has length $\frac{2\sqrt{2}}{\sqrt{2}} = 2$

Fractals

\$100 Which of the following does NOT have zero area?

(a) Sierpinski Triangle

(b) Koch Snowflake

(c) Box Fractal

Answer: (b) Koch Snowflake

\$200 Suppose we have an equilateral triangle with an area of 2 units^2. What is the area of the Sierpinski Triangle after 2 iterations?

The equilateral triangle has an area of 2 units^2. There are $n = 2$ iterations, so the area of the Sierpinski Triangle is

$$A_2 = 2 \times \left(\frac{3}{4}\right)^n = 2 \times \left(\frac{3}{4}\right)^2 = 2 \times \frac{9}{16} = \frac{18}{16} = \frac{9}{8} = 1.125 \text{ units}^2.$$

\$300 Suppose we have an equilateral triangle with side length of 1 unit. What is the perimeter of the Koch Snowflake after 3 iterations?

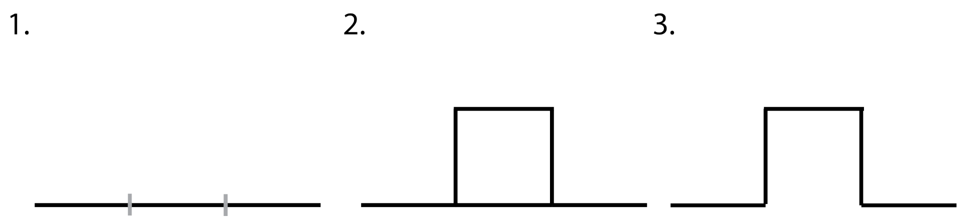
There are $n = 3$ iterations, so the perimeter of the Koch Snowflake is

$$P_3 = 3 \times 4^n \times \left(\frac{1}{3}\right)^n = 3 \times 4^3 \times \left(\frac{1}{3}\right)^3 = 3 \times 64 \times \frac{1}{27} = \frac{192}{27} = \frac{64}{9} = 7.111 \text{ units}.$$

\$400 Here is a variation of the Koch Snowflake.

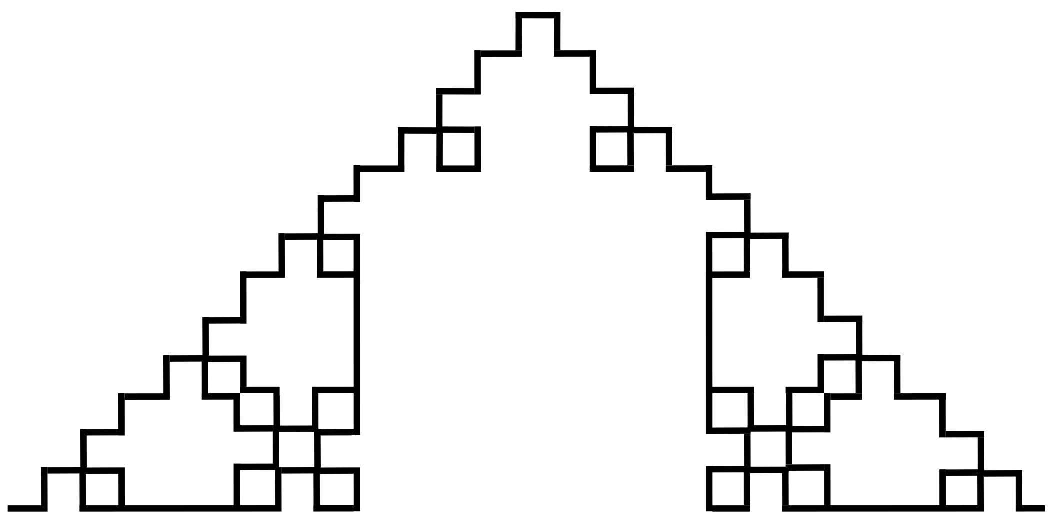
Suppose we start out with a straight line with a length of 9 units.

1. Divide the line into 3 equal line segments
2. In the middle section, using that side length, draw a square with the middle section as one of the sides of the square.
3. Erase the middle section. We are done the first iteration.



4. Repeat steps 1-3 for every line you see for each iteration.

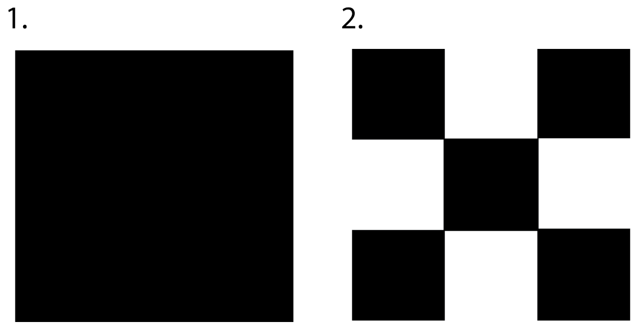
Now, what does the third iteration looks like? What is the total length of the line?
Hint: You could derive a formula, but it isn't necessary in this case.



The side lengths are all of equal length since we continuously divide the line into equal line segments. After each iteration, each line segment is turned into 5 line segments of equal length. Since there are 3 iterations, there are $1 \times 5 \times 5 \times 5 = 5^3 = 125$ equal line segments. The length of each line segment is $\frac{1}{3}$ the length of the previous line segment. Our original line is 9 units long so after 3 iterations, the new length of each line segment is $9 \times \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = 9 \times \left(\frac{1}{3}\right)^3 = 9 \times \frac{1}{27} = \frac{1}{3}$ units. Thus, the total length of the line is $125 \times \frac{1}{3} = 41.67$ units.

\$500 Suppose we have a black square with an area of 27 units^2.

1. Divide the black square into 3 by 3 grid i.e. you should have 9 equal smaller squares within the initial black square.
2. Shade the middle squares on the outside white i.e. you should have 4 white squares as shown below. We are done our first iteration.



3. Repeat steps 1-2 for each black square for subsequent iterations.

Derive a formula to determine the area of all the black squares after the n^{th} iteration.

Notice that after each iteration that remain black. So after each iteration, only $\frac{5}{9}$ of the previous area remains, so each subsequent iteration we multiply by a factor of $\frac{5}{9}$. Let n be the number of iterations. Since, the initial area is 27 units^2. The formula is given below:

$$27 \times \underbrace{\frac{5}{9} \times \frac{5}{9} \times ....\frac{5}{9}}_{\text{n times}} = 27 \times \left(\frac{5}{9}\right)^n$$

Sets

\$100 What is the cardinality of the set $\{12, 3, 4\}$? 3

\$200 How many subsets can you form from a set with 3 elements?

A possible solution is create a set (a, b, c) for reference purposes and then find the number of possible subsets. The possible subsets are as follows:

$$\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$$

We can see by inspection that there are 8 possible subsets.

Alternatively, we can use the formula $2^3 = 8$.

\$300 Given a set with $A = \{2, 3, 6\}$ and $B = \{5, 6, 2, 3\}$ and $\mathbb{U} = \{1, 2, 3, 4, 5, 6\}$. Determine $\overline{\bar{A} \cup B}$.

$A = \{2, 3, 6\} \Rightarrow \bar{A} = \{1, 4, 5\}$
Since $\bar{A} = \{1, 4, 5\}$, then $\bar{A} \cup B = \{1, 2, 3, 4, 5, 6\}$
$\bar{A} \cup B = \{1, 2, 3, 4, 5, 6\} \Rightarrow \overline{\bar{A} \cup B} = \{\} = \emptyset$
Thus, $\overline{\bar{A} \cup B} = \emptyset$.

\$400 There are 120 grade 7/8 math circles registered across the 4 sections on Tuesday and Wednesday nights. During registration, 74 students preferred to be part of the Tuesday section and 65 students wanted to be part of the Wednesday section. How many students didn't care on whether they were in the Tuesday or Wednesday section?

We will use the principle of inclusion and exclusion. Let T be the set of students who prefer the Tuesday session.
Let S be the set of students who prefer the Wednesday session.

By the principle of inclusion and exclusion, we have that

$$\begin{aligned}
|S \cup T| &= |S| + |T| - |S \cap T| \\
|S \cap T| &= |S| + |T| - |S \cup T| \\
&= 74 + 65 - 120 \\
&= 19
\end{aligned}$$

$\therefore$, there are 19 students who did not care whether or not they were in the Tuesday or Wednesday session.

\$500 Recall that a number is a perfect square if it can be expressed as a product of two of the same integers. Let $P = \{1, 4, 9, 16...\}$ be the set of perfect squares. Let $O = \{1, 3, 5, 7, ...\}$ i.e. the set of all odd natural numbers. Show using a rule that the two sets have the same cardinality.

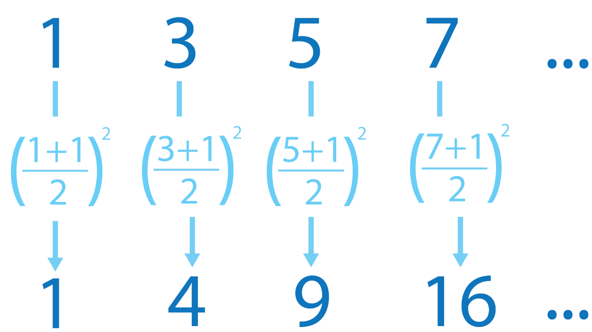
To show that two sets of are the same size, we need to construct a bijection or a rule to match every element in P to every element in O uniquely. The rule is as follows, take any element in O say n: add 1, divide by 2, and then square it. Take 3 for example.

$$\left(\frac{3+1}{2}\right)^2 = 4$$

In general we have that

$$\left(\frac{n+1}{2}\right)^2$$

We can see using the rule, we have matched every element in $P = \{1, 4, 9, 16...\}$ to an element $O = \{1, 3, 5, 7, ...\}$.



Random Questions I

\$100 Hermela doubled a number and then added 4 to get 42. If she had first added 4 to the original number and then doubled the result, what would the answer have been?

We need to determine the value of the original number. So we have

$$42 - 4 = 38 \Rightarrow 38 \div 2 = 19$$

The original number is 19. Now, if we add 4 first and then double the result, we get

$$19 + 4 = 23 \Rightarrow 23 \times 2 = 46$$

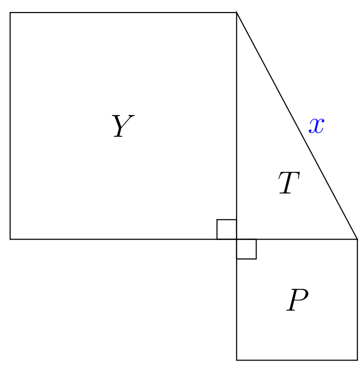
\$200 What is the smallest positive integer by which 54 can be multiplied so that the result is a perfect square?

The prime factorization of 54 is $54 = 2 \times 3 \times 3 \times 3$. Each prime factor has an odd multiplicity. To make a perfect square, the multiplicity of each prime factor of 54 must be even. We can multiply by 2 and 3 to get...

$$54 \times 2 \times 3 = 2 \times 3 \times 3 \times 3 \times 2 \times 3$$

$\therefore$ the smallest positive integer to multiply 54 with to get a perfect square is $2 \times 3 = 6$.

\$300 The perimeter of square Y is 60 cm and the perimeter of square P is 32 cm. What is the perimeter of right-angled triangle T?



The side length of square Y is $60 \div 4 = 15$ cm and the side length of square P is $32 \div 4 = 8$ cm. Since T is a right triangle, by Pythagorean Theorem, the missing side length, x, is $x^2 = 15^2 + 8^2 = 289 \Rightarrow x = 17$ cm.

$\therefore$ the perimeter of T is $15 + 8 + 17 = 40$ cm.

\$400 An integer is chosen at random from 1 to 60 inclusive. What is the probability the integer chosen contains the digit 4?

There are 60 integers in total from 1 to 60 inclusive. There are 6 integers where the ones digits is 4, 4, 14, 24, 34, 44, 54. Then, there are 10 integers where the tens digit is 4, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49. There are $6 + 10 - 1 = 15$ integers in total from 1 to 60 inclusive that contains the digit 4 (we subtract 1 to avoid double counting 44 which is present in both sets).

$\therefore$ the probability that the integer chosen contains the digit 4 is $\frac{15}{60} = \frac{1}{4}$.

\$500 Find the value of $490 - 491 + 492 - 493 + 494 - 495 + \ldots - 509 + 510$.

$490 - 491 + 492 - 493 + 494 - 495 + \ldots - 509 + 510$
$= (490 - 491) + (492 - 493) + (494 - 495) + \ldots + (508 - 509) + 510.$
$= (-1) + (-1) + (-1) + \ldots + (-1) + 510$
$= (-10) + 510$
$= 500$

Gauss Prep

\$100 If $x = -4$ and $y = 4$, which of the following expressions gives the largest answer?

(a) $\frac{x}{y}$ (b) $y - 1$ (c) $x - 1$ (d) $-xy$ (e) $x + y$

By computing each answer, we get -1, 3, -5, 16 and 0 respectively. We see that 16 is the largest answer.

Answer: (d)

\$200 If $2^a = 8$ and $a = 3c$, then c equals

(a) 0 (b) $\frac{3}{4}$ (c) 1 (d) $\frac{4}{3}$ (e) 6

Since $2^3 = 8$ and $2^a = 8$, then $a = 3$. Since $a = 3$ and $a = 3c$, then $c = 1$.

Answer: (c)

\$300 Last year, Kiril's age was a multiple of 7. This year, Kiril's age is a multiple of 5. In how many years will Kiril be 26 years old?

(a) 11 (b) 21 (c) 4 (d) 18 (e) 16

We need to find two consecutive numbers, the first of which is a multiple of 7 and the second of which is a multiple of 5. We try multiples of 7 and the numbers after each.
Do 7 and 8 work? No, since 8 is not a multiple of 5.
Do 14 and 15 work? Yes, since 15 is a multiple of 5.
This means that this year, Kiril is 15 years old. So, in 11 years, Kiril will be 26 years old.

Answer: (a)

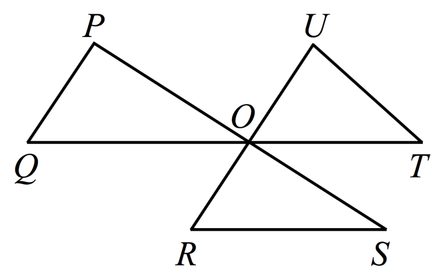
\$400 Andrea has finished the third day of a six-day canoe trip. If she has completed $\frac{3}{7}$ of the trip's total distance of 168 km, how many km per day must she average for the remainder of her trip?

(a) 29 (b) 24 (c) 27 (d) 32 (e) 26

Since Andrea has completed $\frac{3}{7}$ of the total 168 km, then she has completed $\frac{3}{7} \times 168$ km or $3 \times 24 = 72$ km. This means that she has $168 - 72 = 96$ km remaining. To complete the 96 km in her 3 remaining days, she must average $\frac{96}{3} = 32$ km per day.

Answer: (d)

\$500 Lines PS, QT and RU intersect at a common point O,as shown. P is joined to Q, R to S, and T to U, to form triangles. The value of $\angle P + \angle Q + \angle R + \angle S + \angle T + \angle U$ is



(a) 450° (b) 270° (c) 360° (d) 540° (e) 720°

We know that $\angle POQ + \angle POU + \angle UOT = 180°$ because when added they form a straight line. Since $\angle POU = \angle ROS$ (vertically opposite angles), therefore $\angle POQ + \angle ROS + \angle UOT = 180°$. Thus the sum of the remaining angles is just $3 \times 180° - 180° = 360°$ since there are 180° in each of the three given triangles.

Answer: (c)

Game Theory

\$200 What is the equilibrium of the game?

		Player 2	
		X	Y
Player 1	A	$(10, 10)$	$(15, 5)$
	B	$(5, 15)$	$(12, 12)$

The Nash equilibrium is at A,X

\$400 How many equilibria (in pure strategies) does the game below have?

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	$(0, 0)$	$(-1, 1)$	$(1, -1)$
	Paper	$(1, -1)$	$(0, 0)$	$(-1, 1)$
	Scissors	$(-1, 1)$	$(1, -1)$	$(0, 0)$

There are **3** Nash equilibria.

\$600 **Game of Chicken** Once a breakthrough is made by any one company, it quickly becomes an open standard for other companies to adopt. As a result, companies can benefit from other companies investing or researching into emerging technologies because they themselves don't have to. Suppose Apple and Samsung both want to make a product that introduces a breakthrough into artificial intelligence.

1. If both Apple and Samsung invests into A.I, they will both earn 300 million dollars annually .
2. If ONLY Apple invests into A.I, Apple and Samsung will earn 500 million dollars and 100 million dollars per annum respectively.
3. If ONLY Samsung invests into A.I, Apple and Samsung will earn 100 million dollars and 500 million dollars per annum respectively.
4. If neither invests, both companies will not earn anything.

Draw the payoff matrix of the situation and what is the Nash Equilibrium?

		Samsung	
		Invests	Does not Invest
Apple	Invests	$(300, 300)$	$(500, 100)$
	Does Not Invest	$(100, 500)$	$(0, 0)$

There are two Nash equilibrium.

\$800 Kingeon and Caila decided to fundraise for a roadtrip. They can either sell bags of chips or cans of pop.

1. If Kingeon and Caila both sell cans of pop, then Kingeon will sell 35 cans of pop and Caila will sell 50 cans of pop.
2. If Kingeon and Caila both sell bags of chips, then Kingeon will sell 30 bags of chips and Caila will sell 50 bags of chips.
3. If Kingeon sells cans of pop and Caila sells bags of chips, then Kingeon will sell 50 cans of pop and Caila will sell 30 bags of chips.
4. If Kingeon sells chips and Caila sells cans of pop, then Kingeon will sell 30 bags of chips and Caila will sell 70 cans of pop.

Is this a fair competition? If not, what can Kingeon do to ensure that their fundraiser is as successful as possible?

		Caila	
		Pop	Chips
Kingeon	Pop	(35, 50)	(50, 30)
	Chips	(30, 70)	(30, 50)

If we make a payoff matrix of the situation, we see that this is NOT a fair competition. Caila will always have the best strategy no matter what Kingeon chooses to sell. To ensure that the fundraiser is as successful as possible, Kingeon should choose to sell bags of chips while Caila sells cans of pop because together they will sell 100 cans of pop and bags of chips in total which is more items sold than the other possible outcomes (assuming the price of a can of pop is the same as a bag of chips).

$1000 Natasha and Steve are playing a game of 6. In this game, there are 6 sticks in a pile. Each player can take away 1, 2 or 5 sticks on their turn. Whichever player picks up the last stick wins the game. If Natasha is the first player, who will win the game?

Let's take another look at this game and see who wins if there are 1, 2, 3, 4, 5 and 6 sticks in a pile.

n sticks in a pile	Natasha	Steve
1	Win	-
2	Win	-
3	-	Win
4	Win	-
5	Win	-
6	-	Win

If there is 1 stick, Natasha will pick up the stick and win.
If there are 2 sticks, Natasha will pick up both sticks and win.
If there are 3 sticks, Steve will win (because Natasha cannot pick up 5 sticks and if she picks up 1 or 2 sticks, Steve will pick up 2 or 1 stick(s) respectively and win).
If there are 4 sticks, Natasha will pick up 1 stick so Steve is left with 3 sticks. Then if Steve picks up 1 or 2 sticks, Natasha will pick 2 or 1 stick(s) respectively and win.
If there are 5 sticks, Natasha will pick up 2 sticks so Steve is left with 3 sticks. Then if Steve picks up 1 or 2 sticks, Natasha will pick 2 or 1 stick(s) respectively and win.
If there are 6 sticks, Steve will win because ...

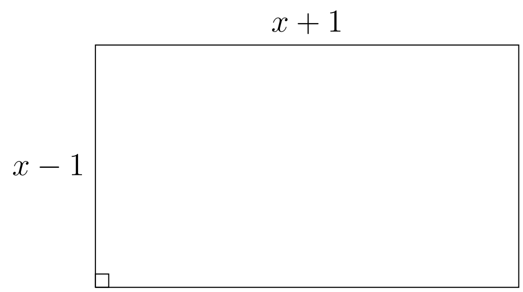
- ... if Natasha picks up 5 sticks, Steve will pick up the last stick and win
- ... if Natasha picks up 2 sticks, Steve will pick up 1 stick leaving Natasha with 3 sticks. Then if Natasha picks up 1 or 2 sticks, Steve will pick 2 or 1 stick(s) respectively and win
- ... if Natasha picks up 1 stick, Steve will pick up 2 sticks leaving Natasha with 3 sticks. Then if Natasha picks up 1 or 2 sticks, Steve will pick 2 or 1 stick(s) respectively and win

Steve has the winning strategy for a game of 6 sticks.

Therefore, Steve will win the game.

Geometry

\$200 If the perimeter of the given rectangle is 24, what is the value of x?

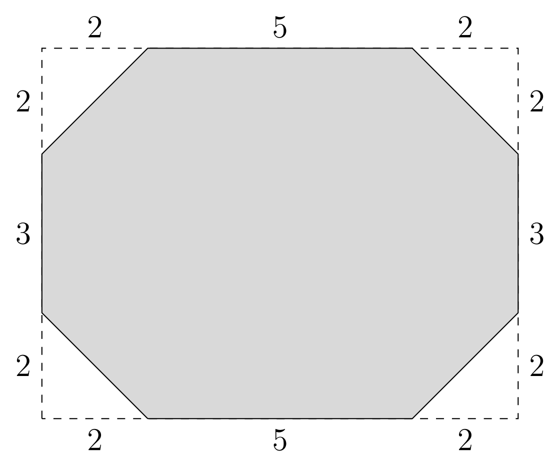


Since the perimeter is 24, we have that...

$$24 = (x+1) + (x+1) + (x-1) + (x-1) \Rightarrow 24 = 4x \Rightarrow x = 6$$

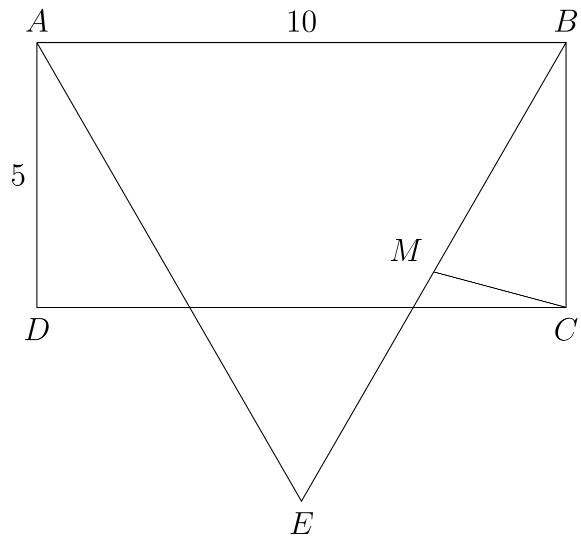
$\therefore x = 6$.

\$400 In the 9 by 7 rectangle shown, what is the area of the shaded region?



The area of the whole rectangle is $9 \times 7 = 63$. Notice that we have four identical triangles at each corner of the rectangle. The sum total of the area of the triangles is $4 \times \dfrac{2 \times 2}{2} = 4 \times 2 = 8$.
$\therefore$ the area of the shaded region is $63 - 8 = 55$.

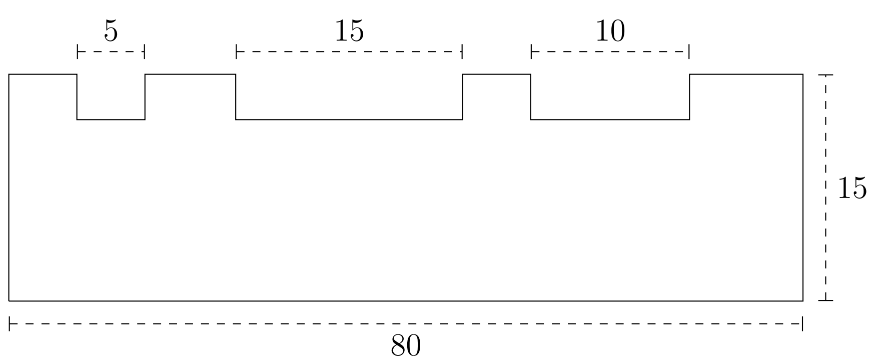
\$600 In the diagram, $ABCD$ is a rectangle and ABE is an equilateral triangle. M is the midpoint of BE. Calculate $\angle BMC$.



Since $ABCD$ is a rectangle and $AD = 5$, then $BC = 5$. Since ABE is an equilateral triangle and $AB = 10$, then $BE = 10$. M is the midpoint of BE which means that $BM = ME = \frac{10}{2} = 5$. Because $BC = 5$ and $BM = 5$, BMC must be an isosceles triangle. Next, since $ABCD$ is a rectangle and ABE is an equilateral triangle, then $\angle ABC = 90°$ and $\angle ABE = 60°$. Then, we have that $\angle CBM = 90° - 60° = 30°$.

Since BMC is isosceles, $\angle BMC = \frac{180°-30°}{2} = \frac{150°}{2} = 75°$.

\$800 Three small rectangles, of the same depth, are cut from a rectangular sheet of metal. The area of the remaining piece is 990. What is the depth of each cut?

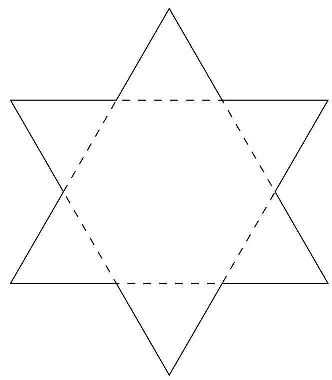


Let x be the depth of the small rectangles cut from the sheet of metal. Then, the areas of the small rectangles are $5x$, $15x$, and $10x$ respectively.
The area of the whole uncut rectangle is $80 \times 15 = 1200$. Since the area of the remaining sheet of metal is 990, the sum total of the areas of the small rectangles must be $1200 - 990 = 210$. And so, we have that...

$$210 = 5x + 15x + 10x \Rightarrow 210 = 30x \Rightarrow x = 7$$

$\therefore$ the depth of each cut is 7.

\$1000 A six-pointed star is formed by extending the edges of a regular hexagon. If the perimeter of the hexagon is 21, what is the perimeter of the star?



Since the perimeter of the regular hexagon is 21, then all the side lengths are equal. Each side length of the hexagon is $\frac{21}{6} = \frac{7}{2}$. The sum of interior angles of a hexagon is $720°$ so each interior angle is $120°$. The supplementary angle of $120°$ is $60°$. This means the two angles in the smaller triangles that form a straight line are $60°$. The missing angle in the triangle is $180° - 60° - 60° = 60°$. Each angle in the smaller triangles is $60°$ so the small triangles are equilateral so the side lengths are all the same meaning we have six identical triangles. The side length of each small triangle is $\frac{7}{2}$ and the perimeter consists of 12 equal side lengths. Thus, the perimeter of the star is $12 \times \frac{7}{2} = 42$.

Combinatorial Counting

\$200 A fog horn sounds a blast for 2 seconds and then is silent for 8 seconds. How many blasts are made in a $3\frac{1}{2}$ hour period?

From the beginning of a blast until the beginning of the next blast is a period of 10 seconds. In $3\frac{1}{2}$ hours, there are $\frac{7}{2} \times 60 \times 60 = 12600$ seconds. Therefore, the number of blasts is $\frac{12600}{10} = 1260$.

\$400 In a class of 30 students, twelve played in the band, and seventeen played volleyball. Of these, five students did both. How many students did not participate in either activity?

Since there are five students who played both in the band and volleyball, the number of students in either one or both activities was $12 + 17 - 5 = 24$. The number of students in the class who did not participate in either of these two activities was $30 - 24 = 6$.

\$600 In how many ways can we arrange the letters in MARCH?

There are 5 letters in MARCH so there are $5! = 120$ ways to arrange the letters.

\$800 In how many ways can we arrange the letters in BANANA?

There are 6 letters in BANANA including three As and two Ns. Thus, there are 6! ways to arrange the letters. However, we have over counted the number of arrangements because of the three As and two Ns. There are also 3! ways to arrange the three As and 2! ways to arrange the two Ns. So, there are $\frac{6!}{3!2!} = 60$ ways to arrange the letters in BANANA.

\$1000 Solve for n in the following: ${}_nC_2 = 66$

$$\begin{aligned}
66 &= {}_nC_2 \\
66 &= \binom{n}{2} \\
2\times\ 66 &= \frac{n!}{2!(n-2)!}\times 2 \\
132 &= \frac{n \times (n-1) \times (n-2)!}{(n-2)!} \\
132 &= n \times (n-1)
\end{aligned}$$

We need two consecutive numbers that multiply to give us 132. Notice that $11^2 = 121$. $121 < 132$ so we know that $n \neq 11$ and that n must be greater than 11. If we try $n = 12$, we get $12 \times 11 = 132$.

Therefore, $n = 12$.

Logic

$200 If Monica's daughter is my daughter's mother, who am I to Monica?

I am Monica's daughter.

$400 At a family reunion of 12 people, the official photographer takes pictures of two people at a time. If each person has his picture taken with each of the other people, determine the maximum number of pictures that could be taken.

The first person will have their picture taken with 11 others. The second person with 10 others, the third person with 9 others, so on and so forth. So we have,

$$11 + 10 + 9 + \ldots + 2 + 1$$

We can use Gauss' sum formula! The sum from 1 to 11 is $\frac{11\times12}{2} = 66$. Thus, the number of pictures taken is 66.

$600 There is a box full of fruits.
All but two are mangoes.
All but two are apples.
All but two are oranges.
How many fruits are in the box?

There are 3 fruits! (One mango, one apple and one orange.)

$800 Using any math symbols, arrange four 9s so that they equal 100.

$$99 + \frac{9}{9} = 100$$

\$1000 Ana, Bernie, Carmelo, Daniel and Elle are all in the same house.

1. If Ana is watching a movie, so is Bernie.
2. Either Daniel, Elle, or both of them are watching a movie.
3. Either Bernie or Carmelo, but not both are watching a movie.
4. Daniel and Carmelo are either both watching or both not watching a movie.
5. If Elle is watching a movie, then Ana and Daniel are also watching a movie.

Who is watching a movie and who is not?

We will let A, B, C, D and E represent Ana, Bernie, Carmelo, Daniel and Elle.

There are a few possible solutions. Below is one such solution:

If E is not watching, then D is watching (by statement 2). By statement 4, C must be watching since D is watching a movie. Since C is watching a movie, B is not watching a movie (by statement 3). By statement 1, A is not watching because B is not watching a movie. Since E is not watching, A and E are also not watching which satisfies statement 5.

Thus, only C and D are watching a movie while A, B and E are not.

Random Questions II

\$200 Which of the following numbers are greater than its square?

$$-20, -\frac{3}{4}, 0, \frac{1}{2}, 10$$

$$\begin{aligned}
(-20)^2 &= 400 > 20 \\
\left(-\frac{3}{4}\right)^2 &= \frac{9}{16} > -\frac{3}{4} \\
0^2 &= 0 = 0 \\
\left(\frac{1}{2}\right)^2 &= \frac{1}{4} < \frac{1}{2} \\
10^2 &= 100 > 10
\end{aligned}$$

$\therefore \frac{1}{2}$ is the only number that is greater than its square.

\$400 My rich uncle gave me \$1 for my first birthday. On each birthday after that, he doubled his previous gift. Find the total amount that he had given me by the day after my 8^{th} birthday.

My gift is multiplied by 2 each year for 8 years so for 8 years, I received, \$1, \$2, \$4, \$8, \$16, \$32, \$64 and \$128. In total, he had given me $\$1 + \$2 + \$4 + \$8 + \$16 + \$32 + \$64 + \$128 = \$255$ by the day after my 8^{th} birthday.

\$600 There are 15 Blue Jays and 14 Orioles perched in 3 trees. Each tree has at least 4 Blue Jays and 2 Orioles. If no tree has more Orioles than Blue Jays, then what is the largest number of birds that can be in one tree?

Since each of the 3 trees contains at least 4 Blue Jays and 2 Orioles, 12 Blue Jays and 6 Orioles have been perched, leaving 3 Blue Jays and 8 Orioles. To produce the greatest number in a tree, as many as possible of the remaining birds should be put in one particular tree. If the 3 remaining Blue Jays are put in one tree, the number of Blue Jays in that tree is now 7. Since no tree may hold more Orioles than Blue Jays, 5 of the remaining Orioles may perch here. The number of birds in the tree is 14. The remaining Orioles may be placed in the remaining trees. The maximum number of birds in one tree is 14.

\$800 How many numbers in the set $\{10, 11, 12, ..., 99\}$ increase in value when the order of their digits is reversed?

From 10 to 19, there will be 8 increases.
From 20 to 29, there will be 7 increases.
From 30 to 39, there will be 6 increases, and so on.
Thus, there will be $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ numbers that increase in value when their digits are reversed.

\$1000 Given $xy = \frac{x}{y} = x - y$, where $y \neq 0$, determine the value of $x + y$.

We have 3 equations:

(1) $xy = \frac{x}{y}$ (2) $xy = x - y$, and (3) $\frac{x}{y} = x - y$.

Note that since $y \neq 0$, then, from (2), $x \neq 0$.

From (1), $xy^2 = x \Rightarrow y^2 = 1 \Rightarrow y = 1$ or $y = -1$.

If $y = 1$, then (2) becomes $x = x - 1$ which is impossible and so $y = -1$.

Thus (2) and (3) both become $-x = x + 1$ and so $x = -\frac{1}{2}$. Thus, $x + y = -\frac{3}{2}$.

Gauss Prep

\$200 The value of $(1+2)^2-(1^2+2^2)$ is

(a) 14 (b) 4 (c) 2 (d) 12 (e) 1

$$\begin{aligned}(1+2)^2-(1^2+2^2) &= 3^2-(1+4)\\ &= 9-5\\ &= 4\end{aligned}$$

Answer: (b)

\$400 If the mean (average) of five consecutive integers is 21, the smallest of the five integers is

(a) 17 (b) 21 (c) 1 (d) 18 (e) 19

The mean of five consecutive integers is equal to the number in the middle. Since the numbers have a mean of 21, if we were to distribute the quantities equally, we would 21, 21, 21, 21, and 21.

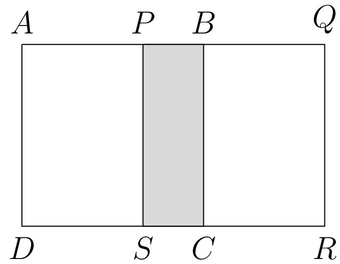
Since the numbers are consecutive, the second number is 1 less than the 21 in the middle, while the fourth number is 1 more than the 21 in the middle.

Similarly, the first number is 2 less than the 21 in the middle, while the fifth number is 2 more than the 21 in the middle.

Thus, the numbers are $21-2, 21-1, 21, 21+1, 21+2$. The smallest of five consecutive integers, having a mean of 21, is 19.

Answer: (e)

\$600 Two identical squares, $ABCD$ and $PQRS$, have side length 12. They overlap to form the 12 by 20 rectangle $AQRD$ shown. What is the area of the shaded rectangle $PBCS$?



(a) 24 (b) 36 (c) 48 (d) 72 (e) 96

Solution 1

Since $AQ = 20$ and $AB = 12$, the $BQ = AQ - AB = 20 - 12 = 8$. Thus, $PB = PQ - BQ = 12 - 8 = 4$. Since $PS = 12$, the area of the shaded rectangle is $12 \times 4 = 48$.

Solution 2

The sum of the areas of squares $ABCD$ and $PQRS$ is $2 \times (12 \times 12) = 1 \times 144 = 288$. The area of rectangle $AQRD$ is equal to the sum of the areas of $APSD, PBCS, PBCS$, and $BQRC$. The area of rectangle $AQRD$ is equal to the sum of the areas of $APSD, PBCS$, and $BQRC$. Therefore, the sum of the areas of $ABCD$ and $PQRS$, minus the areas of $AQRD$, is the area of $PBCS$. Thus, the shaded rectangle $PBCS$ has area $288 - 240 = 48$.

Answer: (c)

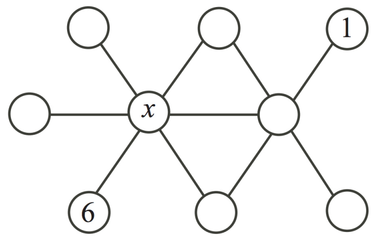
\$800 At the Gaussland Olympics, there are 480 student participants. Each student is participating in 4 different events. Each event has 20 students participating and is supervised by 1 adult coach. There are 16 adult coaches and each coach supervises the same number of events. How many events does each coach supervise?

(a) 12 (b) 8 (c) 6 (d) 16 (e) 15

Since there are 480 student participants and each student is participating in 4 events, then across all events the total number of (non-unique) participants is $480 \times 4 = 1920$. Each event has 20 students participating. Thus, the number of different events is $\frac{1920}{20} = 96$. Each event is supervised by 1 adult coach, and there are 16 adult coaches each supervising the same number of events. Therefore, the number of events supervised by each coach is $\frac{96}{16} = 6$.

Answer: (c)

\$1000 In the diagram, each of the integers 1 through 9 is to be placed in one circle so that the integers in every straight row of three joined circles add to 18. The 6 and 1 have been filled in. The value of the number represented by x is



(a) 4 (b) 5 (c) 8 (d) 7 (e) 3

Consider the possible ways that the numbers from 1 to 9 can be used three at a time to sum to 18. Since we may not repeat any digit in the sum, the possibilities are:

$$1+8+9, 2+7+9, 3+6+9, 3+7+8, 4+5+9, 4+6+8, 5+6+7$$

Next, consider the row $1+d+f$ in Fig. 1 below.

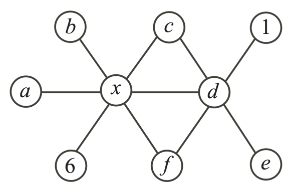


Fig.1

Since the sum of every row is 18, $d+f=17$. This gives that either $d=8$ and $f=9$ or $d=9$ and $f=8$. Next, consider the 3 rows in which x appears. The sums of these 3 rows are $a+x+d, b+x+f$ and $c+x+6$. That is, x appears in exactly 3 distinct sums. Searching our list of possible sums above, we observe that only the numbers 6, 7 and 8 appear in 3 distinct sums. That is, x must equal either 6, 7 or 8. However, the number 6 already appears in the table. Thus, x is not 6. Similarly, we already concluded that either d or f must equal 8. Thus, x is not 8. Therefore, $x=7$ is the only possibility. Fig. 2 shows the completed table and verifies that the number represented by x is indeed 7.

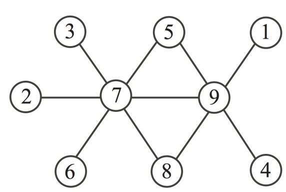
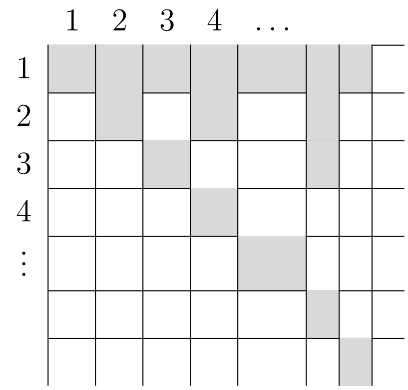


Fig.2

Answer: (d)

Final Jeopardy

In the diagram, the grid has 150 rows and 150 columns, numbered from 1 to 150. In row 1, every box is shaded. In row 2, every second box is shaded. In row 3, every third box is shaded. The shading continues in this way, so that every n-th box in row n is shaded. Which *column* has the greatest number of shaded boxes?



(a) 144 (b) 120 (c) 150 (d) 96 (e) 100

In the 1st row, every box is shaded. In the 2nd row, the boxes whose column numbers are multiples of 2 are shaded. In the 3rd row, the boxes whose columns are multiples of 3 are shaded. In the n-th row, the boxes whose column numbers are multiples of n are shaded. So in a particular column, the boxes which are shaded are those which belong to row numbers which are factors of the column number. (We can see this for instance in columns 4 and 6 where the boxes in rows 1, 2 and 4, and 1, 2, 3, and 6, respectively are shaded.) So to determine which of the given columns has the largest number of shaded boxes, we must determine which of the given numbers has the greatest number of factors.

To find the number of factors of 144, we find that the prime factorization is $144 = 2^4 \times 3^2$ and so there are $(4 + 1) \times (2 + 1) = 5 \times 3 = 15$ factors of 144.

Similarly, we find that . . .

- $120 = 2^3 \times 3 \times 5$, so there are $4 \times 2 \times 2 = 16$ factors of 120
- $150 = 2 \times 3 \times 5^2$, so there are $2 \times 2 \times 3 = 12$ factors of 150
- $96 = 2^5 \times 3$, so there are $6 \times 2 = 12$ factors of 96, and
- $100 = 2^2 \times 5^2$, so there are $3 \times 3 = 9$ factors of 100

The number with the most factors (and thus the column with the most shaded boxes) is 120.

Answer: (b)