Intermediate Math Circles
November 1, 2017
Probability I

Probability is the study of uncertain events or outcomes. Games of chance that involve rolling dice or dealing cards are one obvious area of application. We can apply probability models to study many other phenomena such as the number of vehicles passing through an intersection in a ten minute period, or the length of time it takes to search the internet on a topic.

In each of these situations, the result is uncertain. We will not actually know the result until after we collect the data. In all of these situations, we can use probability models to describe the patterns of variation we are likely to see.

Experiments, Sample Spaces and Events

- An *experiment* is any process that gives rise to data.
- We call the result of the experiment an *outcome*.
- The *sample space* is the set of all possible outcomes of an experiment.
- An *event* is a subset of a sample space. Another way to think of this is that an event is a collection of outcomes that share some property.

The main features of experiments is that there is more than one possible outcome, and the experiments are repeatable. Also, when we repeat the experiment we will not necessarily get the same outcome.

Example 1:

a) Experiment: Toss a fair coin and observe the top face.

Sample Space: $S = \{H, T\}$

Example of Event: $E = \{H\}$

b) Experiment: Roll a die and observe the top face.

Sample Space: $S = \{1, 2, 3, 4, 5, 6\}$

Example of Event: $E = \{1, 3, 5\}$

c) Experiment: Toss a fair coin twice and observe the top face on each trial.

Sample Space: $S = \{HH, HT, TH, TT\}$

Example of Event: $E = \{HT, TH\}$

d) Experiment: Toss a fair coin three times and count the number of heads.

Sample Space: $S = \{0, 1, 2, 3\}$

Example of Event: $E = \{2, 3\}$

A sample space is *discrete* if it contains a finite or countably infinite number of outcomes. The sample spaces in our examples are discrete sample spaces.

Probability Using Counting Techniques

Since we are only going to be considering sample spaces that have a finite number of events, most of our probabilities will come from counting the number of outcomes of a certain type, and dividing this by the total number of possible outcomes.

Theoretical Probability
The probability of an event, E, occurring (where each outcome is equally likely) can be given by:

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ is the number of outcomes in which event E can occur, and $n(S)$ is the total number of outcomes.

This is where the tools from combinatorics come in handy.

Example 2:

Suppose a single card is chosen from a standard 52 card deck. Give the probability of each of the following events.

a) a heart is chosen

b) a face card is chosen

Solution:

a) A standard deck of 52 cards has four suits: hearts, clubs, diamonds and spaces, with 13 cards in each suit.

$$\begin{aligned} P(\text{heart}) &= \frac{n(\text{heart})}{n(\text{any card})} \\ &= \frac{13}{52} \\ &= \frac{1}{4} \end{aligned}$$

b) There are four suits and each suit has a jack, a queen and a king, so there are 12 face cards in the deck.

$$\begin{aligned} P(\text{face}) &= \frac{n(\text{face})}{n(\text{any card})} \\ &= \frac{12}{52} \\ &= \frac{3}{13} \end{aligned}$$

Example 3:

What is the probability of being dealt exactly 2 aces in a 4 card hand?

Solution:

The total number of hands containing 4 cards is:

$$\begin{aligned}\binom{52}{4} &= \frac{52!}{48!4!}\\ &= \frac{\overset{13}{\cancel{52}} \times \overset{17}{\cancel{51}} \times \overset{25}{\cancel{50}} \times 49 \times \cancel{48!}}{\cancel{48!} \times \cancel{4} \times \cancel{3} \times \cancel{2}}\\ &= 13 \times 17 \times 25 \times 49\end{aligned}$$

The number of hands that contain exactly 2 aces is:

$$\begin{aligned}\binom{4}{2} \times \binom{48}{2} &= \frac{4!}{2!2!} \times \frac{48!}{46!2!}\\ &= \frac{\overset{2}{\cancel{4}} \times 3 \times \cancel{2!}}{\cancel{2!} \times \cancel{2}} \times \frac{\overset{24}{\cancel{48}} \times 47 \times \cancel{46!}}{\cancel{46!} \times \cancel{2}}\\ &= 6 \times 24 \times 47\end{aligned}$$

Therefore, the probability of being dealt a 4 card hand containing exactly 2 aces is:

$$\begin{aligned}P(\text{exactly 2 ace cards}) &= \frac{n(\text{exactly 2 ace cards})}{n(\text{any 4 cards})}\\ &= \frac{6 \times 24 \times 47}{13 \times 17 \times 25 \times 49}\\ &= \frac{6768}{270\ 725}\\ &\doteq 0.025\end{aligned}$$

Example 4:

David enters a pie eating contest where five people of equal ability compete. Prizes are awarded to the people that finish first and second. What is the probability that David does not receive a prize?

Solution:

Let D represent the event that David does not receive a prize. If David does not receive a prize then he finished 3rd, 4th or 5th. In each of these cases, there are 4! ways to arrange the other contestants. Therefore, $n(D) = 3 \times 4!$.
Let S represent all the possible orders in which the 5! contestants can finish. Then $n(S) = 5!$ since this is the number of orders in which 5 contestants can finish the contest.

$$\begin{aligned}P(D) &= \frac{n(D)}{n(S)}\\ &= \frac{3 \times 4!}{5!}\\ &= \frac{3}{5}\end{aligned}$$

Therefore the probability that David does not win a prize is $\frac{3}{5}$.

A useful and important concept in probability is the complement of an event. The *complement* of event E, denoted E' (or $\overline{E}$ or E^C), is the event that "event E does not happen". Since E and E' together include all possible outcomes,

$$P(E) + P(E') = 1$$

Example 5:

What is the probability that at least two out of a group of 8 friends will have the same birthday? Assume no student was born on February 29.

Solution:

The simplest method is to consider the complementary event that no two people have the same birthday. Let E represent the event that least 2 students share a birthday, then E' represents the complement, or the event that no two students share a birthday.

There are 365 days in a year. If all 8 students have different birthdays, then:

$$n(E') = 365 \times 364 \times 363 \times 362 \times 361 \times 360 \times 359 \times 358$$

All possible ways 8 students can have 8 birthdays is

$$n(S) = 365 \times 365 \times 365 \times 365 \times 365 \times 365 \times 365 \times 365 = 365^8$$

The probability that all 8 students have a different birthday is:

$$\begin{aligned} P(E') &= \frac{n(E')}{n(S)} \\ &= \frac{\cancel{365} \times 364 \times 363 \times 362 \times 361 \times 360 \times 359 \times 358}{\cancel{365} \times 365 \times 365 \times 365 \times 365 \times 365 \times 365 \times 365} \\ &\doteq 0.926 \end{aligned}$$

Therefore, the probability that at least 2 students share a birthday is:

$$\begin{aligned} P(E) &= 1 - P(E') \\ &= 1 - 0.926 \\ &\doteq 0.074 \end{aligned}$$

Exercises

1. When rolling two fair dice, what is the probability that the sum is:
   a) greater than 5?
   b) less than 2?
   c) equal to 8?
   d) less than or equal to 12?

2. A messy drawer contains five black socks, three brown socks, and eight white socks, none of which are paired up. If two socks are selected without looking, what is the probability that both will be white?

3. A credit card PIN of length 4 is formed by randomly selecting (with replacement) 4 digits from the set $0 - 9$. Find the probability:
   a) the PIN is even
   b) the PIN's digits are all different
   c) the PIN is a palindrome
   d) the digits 2 and 3 occur beside each other

4. What is the probability of there being at least 1 king in a 5 card hand?

Answers

1. a) $P(\text{sum} > 5) = \dfrac{13}{18}$
   b) $P(\text{sum} < 2) = 0$
   c) $P(\text{sum} = 8) = \dfrac{5}{36}$
   d) $P(\text{sum} \leq 12) = 1$

2. $P(\text{2 white socks}) = \dfrac{7}{30}$

3. a) $P(\text{even}) = \dfrac{1}{2}$
   b) $P(\text{different digits}) = \dfrac{63}{125}$
   c) $P(\text{palindrome}) = \dfrac{1}{100}$
   d) $P(\text{contains 23 or 32}) = \dfrac{3}{50}$

4. $P(\text{at least 1 king}) = 1 - \dfrac{\binom{48}{5}}{\binom{52}{5}} \doteq 0.341$

Probability Distributions for Discrete Sample Spaces

We often call the individual possible outcomes of an experiment *points*.

Within a discrete sample space, S, we can arrange the outcomes (points) so that one of them is first, another second, etc. So let the points in the sample space be labelled $1, 2, 3, \ldots$. We assign each point i of S a real number, p_i, which is called the probability of the outcome labelled i.

Probability Distribution
A *probability distribution* is any set of p_i's that satisfy

- $0 \leq p_i \leq 1$
- $p_1 + p_2 + p_3 + \ldots = 1$

on a discrete sample space.

Example 6:

A fair die is rolled and we note the number on the top face.
The sample space is: $S = \{1, 2, 3, 4, 5, 6\}$
A reasonable probability distribution would be

$$p_1 = \frac{1}{6}, p_2 = \frac{1}{6}, p_3 = \frac{1}{6}, p_4 = \frac{1}{6}, p_5 = \frac{1}{6}, p_6 = \frac{1}{6}$$

Example 7:

A weighted die is rolled and we note the number on the top face. The die is weighted so that even numbers occur twice as often as odd numbers.
A reasonable probability distribution would be

$$p_1 = \frac{1}{9}, p_2 = \frac{2}{9}, p_3 = \frac{1}{9}, p_4 = \frac{2}{9}, p_5 = \frac{1}{9}, p_6 = \frac{2}{9}$$

Once we have defined a probability distribution for the individual outcomes, we can determine the probability of more complex events.

The Union Rule
To find the probability of the union of two events E and F in a sample space, we use the union rule for counting, which is

$$n(E \cup F) = n(E) + n(F) - n(E \cap F)$$

Union Rule for Probability
For any events E and F from a sample space S,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Example 8:

If a single card is drawn from an ordinary deck of cards, what is the probability that it will be a club or a face card?

Solution:

$$\begin{aligned} P(\text{club} \cup \text{face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club} \cap \text{face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} \\ &= \frac{11}{26} \end{aligned}$$

Example 9:

In a coin toss, what is the probability that heads is flipped at least two times out of three tosses?

Solution:

$$\begin{aligned} P(2H \cup 3H) &= P(2H) + P(3H) - P(2H \cap 3H) \\ &= \frac{3}{8} + \frac{1}{8} \\ &= \frac{4}{8} \\ &= \frac{1}{2} \end{aligned}$$

Note: It is not possible for 2 heads and 3 heads to occur at the same time. These two events are said to be *mutually exclusive*. When two events E and F are mutually exclusive, the union rule can be simplified.

Union Rule for Mutually Exclusive Events
For mutually exclusive events E and F from a sample space S,

$$P(E \cup F) = P(E) + P(F)$$

Example 10:

A die is rolled. What is the probability of rolling an even number?

Solution:

Fair Die

$$\begin{aligned} P(\text{even number}) &= P(2) + P(4) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= \frac{3}{6} \\ &= \frac{1}{2} \end{aligned}$$

Weighted Die from Example 7

$$\begin{aligned} P(\text{even number}) &= P(2) + P(4) + P(6) \\ &= \frac{2}{9} + \frac{2}{9} + \frac{2}{9} \\ &= \frac{6}{9} \\ &= \frac{2}{3} \end{aligned}$$

Example 11:

If two fair dice are rolled, what is the probability that the sum of the numbers showing is less than 11?

Solution:

$$\begin{aligned} P(\text{sum} < 11) &= 1 - P(\text{sum} \geq 11) \\ &= 1 - \Big(P(\text{sum} = 11) + P(\text{sum} = 12)\Big) \\ &= 1 - \left(\frac{2}{36} + \frac{1}{36}\right) \\ &= 1 - \frac{3}{36} \\ &= \frac{33}{36} \\ &= \frac{11}{12} \end{aligned}$$

Probability 1: Problem Set

1. Suppose we draw one card from a well-shuffled deck. Let A be the event that we get a spade, and B be the event we get an ace.

   (a) Are these events mutually exclusive?

   Solution:

   No because it is possible to choose an ace and a spade at the same time, specifically the ace of spades.

   (b) What is the probability of drawing and ace or a spade?

   Solution:

$$\begin{aligned} P(\text{ace} \cup \text{spade}) &= P(\text{ace}) + P(\text{spade}) - P(\text{ace} \cap \text{spade}) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

2. Mark has a bag that contains 3 black marbles, 6 gold marbles, 2 purple marbles, and 6 red marbles. Mark adds a number of white marbles to the bag and tells Susan that if she now draws a marble at random from the bag, the probability of it being black or gold is $\frac{3}{7}$. How many white marbles die Mark add to the bag?

   Solution:

   First we can determine the $P(\text{black or gold}) = P(\text{black} \cup \text{gold})$.

$$\begin{aligned} P(\text{black} \cup \text{gold}) &= P(\text{black}) + P(\text{gold}) \\ &= \frac{n(\text{black})}{n(\text{any marble})} + \frac{n(\text{gold})}{n(\text{any marble})} \\ &= \frac{3}{17} + \frac{6}{17} \\ &= \frac{9}{17} \end{aligned}$$

   After Mark added white marbles, $P(\text{black} \cup \text{gold}) = \frac{3}{7} = \frac{9}{21}$. This means that 4 white marbles were added.

3. In the 6/49 lottery, six different numbers must be selected between 1 through 49 inclusive (order is not important).

a) To win the grand prize your 6 numbers must match the 6 numbers drawn. What is the probability of winning the grand prize?

Solution:

Let A represent the event that your 6 numbers match the 6 numbers drawn. Then

$$n(A) = \binom{6}{6} = 1$$

Let S represent all the possible groups of 6 numbers. Then

$$n(S) = \binom{49}{6} = \frac{49!}{43!\ 6!} = 13\ 983\ 816$$

Therefore, the probability of matching all six numbers is

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{1}{13\ 983\ 816} \end{aligned}$$

b) To win any prize you must have at least 2 numbers that match the 6 numbers drawn. What is the probability of winning any prize?

Solution:

Let B represent that you match at least 2 numbers. Then

$$\begin{aligned} P(B) &= 1 - P(B') \\ &= 1 - P(0 \text{ match} \cup 1 \text{ match}) \\ &= 1 - \Big(P(0 \text{ match}) + P(1 \text{ match})\Big) \\ &= 1 - \left(\frac{\binom{43}{6}}{13\ 983\ 816} + \frac{\binom{6}{1}\binom{43}{5}}{13\ 983\ 816}\right) \\ &= 1 - \left(\frac{6\ 096\ 454}{13\ 983\ 816} + \frac{6 \times 962\ 598}{13\ 983\ 816}\right) \\ &= 1 - \left(\frac{11\ 872\ 042}{13\ 983\ 816}\right) \\ &= 0.151 \end{aligned}$$

Therefore, the probability of winning any prize is approximately 15.1%.

4. (a) What is the probability of being dealt 1 kings in a 4 card hand?

Solution:

Here we have to count all possible ways to have exactly 1 king in our 4 cards. To do this, we break up the deck into 2 parts: kings and non-kings. Of the 4 kings in a deck, we need exactly 1 of them, the number of ways we can choose 1 king is given by $\binom{4}{1}$. Since we want exactly 1 king, the other 3 cards must be non-kings. The number of choices here is given be $\binom{48}{3}$. Multiplying these together and dividing by the number of all possible hands we get

$$P(\text{exactly 1 king}) = \frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{4}} = \frac{69\ 184}{270\ 725} \doteq 0.256$$

(b) Write a formula for the probability of being dealt exactly i kings in a 4 card hand.

Solution:

We can generalize the reasoning behind the expression in part a) to write a formula for the probability of being dealt exactly i kings.

- there are $\binom{4}{i}$ ways of choosing i kings
- there are $\binom{48}{4-i}$ ways of choose the remaining cards

Therefore,

$$P(\text{exactly } i \text{ kings}) = \frac{\binom{4}{i}\binom{48}{4-i}}{\binom{52}{4}}, \text{ for } i = 0, 1, 2, 3, 4$$

5. Suppose a die is weighted so that when it is rolled, the probability of seeing any number on the top face is proportional to the number on the face.

(a) Give the probability distribution that would apply.

Solution:

Let a represent the probability of landing on 1. Then, $p_1 = a$.
Since the probability of seeing any number is proportional to the number on the face

- $p_2 = 2a$ because $2 = 2 \times 1$
- $p_3 = 3a$ because $3 = 3 \times 1$
- $p_4 = 4a$ because $4 = 4 \times 1$
- $p_5 = 5a$ because $5 = 5 \times 1$
- $p_6 = 6a$ because $6 = 6 \times 1$

Since $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ is a probability distribution, we have

$$\begin{aligned} p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 1 \\ a + 2a + 3a + 4a + 5a + 6a &= 1 \\ 21a &= 1 \\ a &= \frac{1}{21} \end{aligned}$$

Therefore, the following probability distribution would apply.

$$p_1 = \frac{1}{21}, p_2 = \frac{2}{21}, p_3 = \frac{3}{21}, p_4 = \frac{4}{21}, p_5 = \frac{5}{21}, p_6 = \frac{6}{21}$$

(b) What is the probability of rolling a multiple of 3?

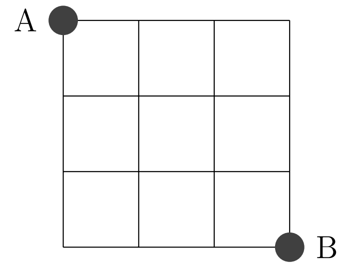
Solution:

The only two faces that contain a multiple of three are 3 and 6.

$$\begin{aligned} P(3 \cup 6) &= P(3) + P(6) \\ &= \frac{3}{21} + \frac{6}{21} \\ &= \frac{9}{21} \\ &= \frac{3}{7} \end{aligned}$$

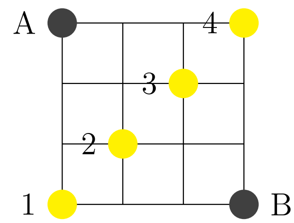
6. A network of paths forms a grid as shown in the following diagram.

Abby starts at point A and walks towards point B. At the same time Bruno starts at point B walks towards point A. Neither person follows a particular route, but they are always moving towards their destination. What is the probability that they will meet if they both walk at the same rate?



Solution:

Since both people are moving at the same rate, if they meet, then they will have to meet at one of the points along the diagonal. These 4 points where Abby and Bruno could meet are circles and labelled $1, 2, 3$, and 4 for reference.



There are a total of $2^3 = 8$ different paths that Abby can take to get to any one of these four points. Similarly, there are $2^3 = 8$ different paths that Bruno can take to get to any one of these four points.

For every path Abby takes, Bruno can take any one of his 8 paths. Therefore, there are $8 \times 8 = 64$ total combinations of paths that result in each person being at one of the four points along the diagonal.

Let's now consider all of the cases where Abby and Bruno meet at the same point.

CASE 1: Abby and Bruno meet at P1.

- Abby has 1 path to this point.
- Bruno has 1 path to this point.

There is $1 \times 1 = 1$ combination of paths that result in Abby and Bruno meeting at P1.

CASE 2: Abby and Bruno meet at P2.

- Abby has 3 paths to this point.
- Bruno has 3 paths to this point.

There is $3 \times 3 = 9$ combinations of paths that result in Abby and Bruno meeting at P2.

CASE 3: Abby and Bruno meet at P3.

- Abby has 3 paths to this point.
- Bruno has 3 paths to this point.

There is $3 \times 3 = 9$ combinations of paths that result in Abby and Bruno meeting at P3.

CASE 4: Abby and Bruno meet at P4.

- Abby has 1 path to this point.
- Bruno has 1 path to this point.

There is $1 \times 1 = 1$ combination of paths that result in Abby and Bruno meeting at P4.

Combining all of these cases, there are a total of $1 + 9 + 9 + 1 = 20$ paths that result in Abby and Bruno meeting.

Thus,

$$\begin{aligned} P(\text{meet}) &= \frac{n(\text{meet})}{n(\text{any path})} \\ &= \frac{20}{64} \\ &= \frac{5}{16} \end{aligned}$$

The probability that Abby and Bruno meet is $\frac{5}{16}$.