Intermediate Math Circles
November 1, 2017
Probability I

Problem Set Solutions

1. Suppose we draw one card from a well-shuffled deck. Let A be the event that we get a spade, and B be the event we get an ace.

   (a) Are these events mutually exclusive?

   Solution:

   No because it is possible to choose an ace and a spade at the same time, specifically the ace of spades.

   (b) What is the probability of drawing and ace or a spade?

   Solution:

$$\begin{aligned} P(\text{ace} \cup \text{spade}) &= P(\text{ace}) + P(\text{spade}) - P(\text{ace} \cap \text{spade}) \\ &= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} \\ &= \frac{16}{52} \\ &= \frac{4}{13} \end{aligned}$$

2. Mark has a bag that contains 3 black marbles, 6 gold marbles, 2 purple marbles, and 6 red marbles. Mark adds a number of white marbles to the bag and tells Susan that if she now draws a marble at random from the bag, the probability of it being black or gold is $\frac{3}{7}$. How many white marbles die Mark add to the bag?

   Solution:

   First we can determine the $P(\text{black or gold}) = P(\text{black} \cup \text{gold})$.

$$\begin{aligned} P(\text{black} \cup \text{gold}) &= P(\text{black}) + P(\text{gold}) \\ &= \frac{n(\text{black})}{n(\text{any marble})} + \frac{n(\text{gold})}{n(\text{any marble})} \\ &= \frac{3}{17} + \frac{6}{17} \\ &= \frac{9}{17} \end{aligned}$$

   After Mark added white marbles, $P(\text{black} \cup \text{gold}) = \frac{3}{7} = \frac{9}{21}$. This means that 4 white marbles were added.

3. In the 6/49 lottery, six different numbers must be selected between 1 through 49 inclusive (order is not important).

a) To win the grand prize your 6 numbers must match the 6 numbers drawn. What is the probability of winning the grand prize?

Solution:

Let A represent the event that your 6 numbers match the 6 numbers drawn. Then

$$n(A) = \binom{6}{6} = 1$$

Let S represent all the possible groups of 6 numbers. Then

$$n(S) = \binom{49}{6} = \frac{49!}{43!\ 6!} = 13\ 983\ 816$$

Therefore, the probability of matching all six numbers is

$$\begin{aligned} P(A) &= \frac{n(A)}{n(S)} \\ &= \frac{1}{13\ 983\ 816} \end{aligned}$$

b) To win any prize you must have at least 2 numbers that match the 6 numbers drawn. What is the probability of winning any prize?

Solution:

Let B represent that you match at least 2 numbers. Then

$$\begin{aligned} P(B) &= 1 - P(B') \\ &= 1 - P(0 \text{ match} \cup 1 \text{ match}) \\ &= 1 - \Big(P(0 \text{ match}) + P(1 \text{ match})\Big) \\ &= 1 - \left(\frac{\binom{43}{6}}{13\ 983\ 816} + \frac{\binom{6}{1}\binom{43}{5}}{13\ 983\ 816}\right) \\ &= 1 - \left(\frac{6\ 096\ 454}{13\ 983\ 816} + \frac{6 \times 962\ 598}{13\ 983\ 816}\right) \\ &= 1 - \left(\frac{11\ 872\ 042}{13\ 983\ 816}\right) \\ &= 0.151 \end{aligned}$$

Therefore, the probability of winning any prize is approximately 15.1%.

4. (a) What is the probability of being dealt 1 kings in a 4 card hand?

<u>Solution:</u>

Here we have to count all possible ways to have exactly 1 king in our 4 cards. To do this, we break up the deck into 2 parts: kings and non-kings. Of the 4 kings in a deck, we need exactly 1 of them, the number of ways we can choose 1 king is given by $\binom{4}{1}$. Since we want exactly 1 king, the other 3 cards must be non-kings. The number of choices here is given be $\binom{48}{3}$. Multiplying these together and dividing by the number of all possible hands we get

$$P(\text{exactly 1 king}) = \frac{\binom{4}{1}\binom{48}{3}}{\binom{52}{4}} = \frac{69\ 184}{270\ 725} \doteq 0.256$$

(b) Write a formula for the probability of being dealt exactly i kings in a 4 card hand.

<u>Solution:</u>

We can generalize the reasoning behind the expression in part a) to write a formula for the probability of being dealt exactly i kings.

- there are $\binom{4}{i}$ ways of choosing i kings
- there are $\binom{48}{4-i}$ ways of choose the remaining cards

Therefore,

$$P(\text{exactly } i \text{ kings}) = \frac{\binom{4}{i}\binom{48}{4-i}}{\binom{52}{4}}, \text{ for } i = 0, 1, 2, 3, 4$$

5. Suppose a die is weighted so that when it is rolled, the probability of seeing any number on the top face is proportional to the number on the face.

   (a) Give the probability distribution that would apply.

   Solution:

   Let a represent the probability of landing on 1. Then, $p_1 = a$.
   Since the probability of seeing any number is proportional to the number on the face

   - $p_2 = 2a$ because $2 = 2 \times 1$
   - $p_3 = 3a$ because $3 = 3 \times 1$
   - $p_4 = 4a$ because $4 = 4 \times 1$
   - $p_5 = 5a$ because $5 = 5 \times 1$
   - $p_6 = 6a$ because $6 = 6 \times 1$

   Since $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ is a probability distribution, we have

$$\begin{aligned} p_1 + p_2 + p_3 + p_4 + p_5 + p_6 &= 1 \\ a + 2a + 3a + 4a + 5a + 6a &= 1 \\ 21a &= 1 \\ a &= \frac{1}{21} \end{aligned}$$

   Therefore, the following probability distribution would apply.

$$p_1 = \frac{1}{21}, p_2 = \frac{2}{21}, p_3 = \frac{3}{21}, p_4 = \frac{4}{21}, p_5 = \frac{5}{21}, p_6 = \frac{6}{21}$$

   (b) What is the probability of rolling a multiple of 3?

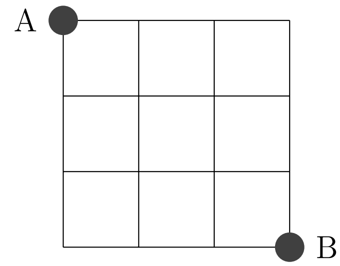
   Solution:

   The only two faces that contain a multiple of three are 3 and 6.

$$\begin{aligned} P(3 \cup 6) &= P(3) + P(6) \\ &= \frac{3}{21} + \frac{6}{21} \\ &= \frac{9}{21} \\ &= \frac{3}{7} \end{aligned}$$

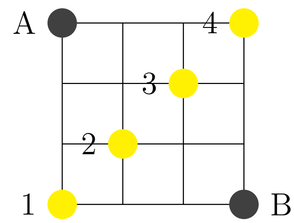
6. A network of paths forms a grid as shown in the following diagram.

Abby starts at point A and walks towards point B. At the same time Bruno starts at point B walks towards point A. Neither person follows a particular route, but they are always moving towards their destination. What is the probability that they will meet if they both walk at the same rate?



Solution:

Since both people are moving at the same rate, if they meet, then they will have to meet at one of the points along the diagonal. These 4 points where Abby and Bruno could meet are circles and labelled $1, 2, 3$, and 4 for reference.



There are a total of $2^3 = 8$ different paths that Abby can take to get to any one of these four points. Similarly, there are $2^3 = 8$ different paths that Bruno can take to get to any one of these four points.

For every path Abby takes, Bruno can take any one of his 8 paths. Therefore, there are $8 \times 8 = 64$ total combinations of paths that result in each person being at one of the four points along the diagonal.

Let's now consider all of the cases where Abby and Bruno meet at the same point.

CASE 1: Abby and Bruno meet at P1.

- Abby has 1 path to this point.
- Bruno has 1 path to this point.

There is $1 \times 1 = 1$ combination of paths that result in Abby and Bruno meeting at P1.

CASE 2: Abby and Bruno meet at P2.

- Abby has 3 paths to this point.
- Bruno has 3 paths to this point.

There is $3 \times 3 = 9$ combinations of paths that result in Abby and Bruno meeting at P2.

CASE 3: Abby and Bruno meet at P3.

- Abby has 3 paths to this point.
- Bruno has 3 paths to this point.

There is $3 \times 3 = 9$ combinations of paths that result in Abby and Bruno meeting at P3.

CASE 4: Abby and Bruno meet at P4.

- Abby has 1 path to this point.
- Bruno has 1 path to this point.

There is $1 \times 1 = 1$ combination of paths that result in Abby and Bruno meeting at P4.

Combining all of these cases, there are a total of $1 + 9 + 9 + 1 = 20$ paths that result in Abby and Bruno meeting.

Thus,

$$\begin{aligned} P(\text{meet}) &= \frac{n(\text{meet})}{n(\text{any path})} \\ &= \frac{20}{64} \\ &= \frac{5}{16} \end{aligned}$$

The probability that Abby and Bruno meet is $\frac{5}{16}$.