## Population Models: Week 1

**Question 1** Consider the population model discussed in the slides

$$P^{n+1} = P^n + (\alpha - \beta)P^n$$

Set $\alpha = 1$ and $\beta = 0.9$. If $P^0 = 100$ compute a table of $P^n$ for $n = 1, ..., 10$. Is there something strange that the math gives as far as the interpretation of $P^n$ as the number of individuals is concerned? (Hint: what does a population of 2.3 mean?)

Solution:

| $n$ | $P^n$ |
|---|---|
| 0 | 100 |
| 1 | 110 |
| 2 | 121 |
| 3 | 133.1 |
| 4 | 146.41 |
| 5 | 161.051 |
| 6 | 177.156 |
| 7 | 194.872 |
| 8 | 214.359 |
| 9 | 235.795 |
| 10 | 259.374 |

The one odd thing that jumps out at you right away is the decimal values. This is natural from a math point of view since a bit of algebra shows that

$$P^{n+1} = P^n(1 + \alpha - \beta) = 1.1P^n$$

so that unless $P^n$ is divisible by ten you must get decimals. Of course it is trickier as far as actual populations since, as the hint says, a population 2.3 is either gruesome (0.3 of an individual) or wrong. The alternative is to accept this as a mathematical artifact of what is otherwise a nice model and accept you just have to round the result to have it make sense.

**Question 2** Consider the population model discussed in the slides

$$P^{n+1} = P^n + (\alpha - \beta)P^n$$

Set $\alpha = 0.9$ and $\beta = 1$. If $P^0 = 100$ compute a table of $P^n$ for $n = 1, ..., 10$. Does the population ever die out? You may need to make a graph.

Solution:

| $n$ | $P^n$ |
|---|---|
| 0 | 100 |
| 1 | 90 |
| 2 | 81 |
| 3 | 72.9 |
| 4 | 65.61 |
| 5 | 59.05 |
| 6 | 53.144 |
| 7 | 47.83 |
| 8 | 43.047 |
| 9 | 38.742 |
| 10 | 34.87 |

You can either graph the above or just do a bit of algebra to notice that

$$P^{n+1} = P^n(1 + \alpha - \beta) = 0.9P^n$$

so that the new value of the population is a multiple less than one of the previous population. That means it is smaller than the previous population, but because the multiple is positive, it also means that the new population will never be zero. Of course the issue of whether the decimal values of population make sense comes back here again, and it is would certainly be reasonable to say that once $P < 1$ we just round that to zero and say that the population has died out.

**Question 3** In Question 1 you identified a flaw in the model, as far as representing the number of individuals. Use the floor function,

$$\lfloor 2.3 \rfloor = 2$$

to come up with a better model and repeat question 1 for this better model.

Solution:

| $n$ | old $P^n$ | new $P^n$ |
|---|---|---|
| 0 | 100 | 100 |
| 1 | 110 | 110 |
| 2 | 121 | 121 |
| 3 | 133.1 | 133 |
| 4 | 146.41 | 146 |
| 5 | 161.051 | 160 |
| 6 | 177.156 | 176 |
| 7 | 194.872 | 193 |
| 8 | 214.359 | 212 |
| 9 | 235.795 | 233 |
| 10 | 259.374 | 256 |

It is fairly easy to fix the model using the floor function (for the record it is also easy to code this up in essentially all mathematical software; like Matlab for example). Wha is more interesting is how the old and new models are similar and how they differ. You can see that steps 0-2 are exactly the same, and steps 3-4 are the same after rounding. But there after the differences are visible. To get some idea as to how big they are we could compute the relative error of the last step where the new model is the "real" answer:

$$E = \frac{259.374 - 256}{256} = 0.0132$$

or about 1% after ten iterations. If you stop and think about it a little bit, the fact that

$$P^{n+1} = \lfloor 1.1 P^n \rfloor$$

means that with each step the error is magnified. To be fair, this population model is super simple, and so any error incurred by not using the floor function is likely very small compared to any number of other more realistic aspects of real world populations that we missed.

**Question 4** Let's try to get more specific and realistic. If we measure time in months and assume all months to be equal we could try to come up with a more realistic model that accounts for harsh winters. Let's say in winter months, which we will take to be December, January, and February we have $\alpha = 0$ and $\beta = 0.25$ and during the summer months we have $\alpha = 0.5$ and $\beta = 0.1$.
i) If we start with $P^0 = 100$ again calculate the population after a year.
ii) Now do some math to try to come up with a simplified model at the end of each season.
iii) Can you find a closed form expression for the population after one year? After n years?

In the slides we mentioned the importance on notation to mathematics (the saying is "if you don't think notation matters, try long division with Roman numerals"). The Julian calendar is a form of notation, and for this problem not a very good one. So let's start our calendar at the start of winter. Thus our initial population is

$$P^0 = 100$$

at the start of December. We have that for the winter months the population follows the rule

$$P^{n+1} = P^n + (0 - 0.25)P^n = 0.75P^n$$

so that after three "winter" months we have

$$P^3 = 0.75P^2 = (0.75)^2 P^1 = (0.75)^3 P^0.$$

Now for the "summer" months we have

$$P^{n+1} = P^n + (0.5 - 0.1)P^n = 1.4P^n$$

so that, skipping a bit of the algebra, after 9 months of "summer" we have

$$P^{12} = (1.4)^9 P^3$$

and substituting our result for winter we have

$$P^{12} = (1.4)^9 (0.75)^3 P^0.$$

Let's refer to the population after n years using the change in notation $\tilde{P}^n$. From our work above

$$\tilde{P}^1 = (1.4)^9 (0.75)^3 100 = 871.64$$

or if we use the floor function as in Question 3, $\tilde{P}^1 = 871$. After 2 years we have

$$\tilde{P}^2 = (1.4)^9 (0.75)^3 \tilde{P}^1 = (1.4)^{18} (0.75)^6 \tilde{P}^0$$

so that after $n$ years we have

$$\tilde{P}^n = (1.4)^{9n} (0.75)^{3n} \tilde{P}^0.$$

This form of the answer explicitly shows the multiplier for the two seasons. This is useful when it comes to critiquing the model, but on a straight up mathematics level we can simplify. With a bit of calculator work, notice

$$(1.4)^9 = (2.744)^3$$

so that

$$(1.4)^{9n}(0.75)^{3n} = (2.744 \times 0.75)^{3n} = (2.058)^{3n}$$

but

$$(2.058)^{3n} = 8.7164^n$$

and finally

$$\tilde{P}^n = (8.7164)^n \tilde{P}^0.$$

Come to think of it we should have just been able to read this off from our solution after one year above! But quite often that's how it goes with mathematical simplifications. You only see what you should have noticed in the first place after laboriously working it out.

Now let's offer a brief critique. This model says that after one your our population have increased more than 8 fold. That may be OK for something like mold growing on bread (though obviously not on the time scale of a year; no one leaves bread out that long), but it is difficult to see how it could be accurate for an animal. Why is that? Well, our intuition suggests that at that rate of increase an animal would use up all its available food and hence there would be a natural check on the population. So the next step would be to either create a better model, or better estimate the parameters of the present model to make it more realistic. To give a concrete example of the latter, let's keep summer as is, but let's make the death rate for the winter $\beta = 0.365$. Working it all out I now get a much ore reasonable multiplier of 1.0047 so that

$$\tilde{P}^n = (1.0047)^n \tilde{P}^0.$$

A bit of trial and error shows that the population doubles after 148 years, by which time surely other aspects of the model need re-examination.