Population Models: Week 2

Question 1 Consider the population model for pairs of goats on an island, with hunters as discussed in the slides

$$P^{n+1} = \lfloor (1+\alpha-\beta)P^n \rfloor - H^n$$

Set $\alpha = 1$ and $\beta = 0.9$. If $P^0 = 100$ compute a table of P^n for $n = 1, ..., 5$ with 2, 4 and 6 hunters. Then get together with three of your neighbours and compare what you got. Did all of you get the same result? What is a sensible way to put all the results together?

Solution:

The results you find should be very similar to what I showed in the slides. Your results should depend on the way your particular coin flips went. Thus, your neighbours' results may not go the same way yours did. To get a general idea of what happens you could average the results your group of 4 got. Recall that the average of 5 random events is the sum of all 5 values divided by 5. Similarly for N events it is a sum of all the events divided by N.

With two hunters my realization gave

n	1st P^n
0	100
1	109
2	119
3	128
4	138
5	150

With 6 hunters I got

n	1st P^n
0	100
1	107
2	115
3	120
4	130
5	143

The trend is suggestive, in that the 6 hunters experiment has a lower value at all times. If I average over a number of experiments I can do better. Since I am not doing this by hand (remember I use Matlab) I did 500 experiments (it took a second) and after 5 years I got an average of $\langle P^5 \rangle = 152.346$ with 2 hunters and an average of 140.16 with 6 hunters. That's a pretty clear difference! Note also that even with the floor function, the mean of several experiments can be a decimal.

Question 2 If $\alpha = \beta$ what does the hunting model predict (you can do this for an arbitrary of hunters or if you'd prefer 5 hunters)? Is there a way to make the prediction more quantitative (HINT: think about what happens on average). Recall that the average of 5 random events is the sum of all 5 values divided by 5. Similarly for N events it is a sum of all the events divided by N.

Solution:

Here you think about the extreme cases. The population model without hunting states that the population remains the same. The hunters never add any new individuals to the population, but they can take some away. Thus the population either decreases or stays the same at each time. For it to not decrease would mean all the hunters would always have to flip the coin correctly to not decrease the population! This means that depending on the way the coin flips go, it may take a longer or shorter amount of time, but no matter what, the hunters will eventually hunt out all the individuals.

Question 3 In the last question what you actually showed is that in a population that undergoes neither net birth or net death, hunting leads to inevitable extinction. Now imagine an alternative scenario in which each year instead of 5 hunters, 5 conservationists visit our island and flip a coin to see if they release a pair of goats or take it back home with them (a little weird I know). Write down the model for this scenario. If $\beta = 1$, $\alpha = 0.95$ carry out an experiment for 5 successive years and again compare with 4 neighbours.

Solution:
The solution for the model is actually presented in the statement of Q4 question.

In terms of examples, we just use the same idea as the hunters but now either add a pair of goats, or add nothing at all. I got:

n	1st P^n	2nd P^n	3rd P^n	4th P^n
100	100	100	100	100
110	113	108	114	112
122	125	121	128	124
130	140	137	140	136
143	156	152	155	147
158	171	169	173	159

Question 4 If we now have a model with 5 hunters (like Question 1) and 5 conservationists (like Question 3) and $\alpha = \beta$ can you show that on average the population stays the same? Do an experiment for 5 years. Does the population stay the same in your experiment? Unless you are particularly lucky, you should find it does not. Can you explain why not? HINT: the model can be written down as:

$$P^{n+1} = \lfloor (1 + \alpha - \beta) P^n \rfloor - H^n + C^n$$

where H^n is the random part due to hunters and C^n the random part due to conservationists.

Solution:

Here the key thing is to notice something about the property of the average, which you may recall we denoted as $\langle f \rangle$. Let's say we have two variables that we get by flipping a coin, rolling a dice or some other random means. We will call them a and b, and since we want to do lots of experiments we denote each time we do the coin flip or die roll as a_j and b_j. If there are K total experiments then

$$\langle a_j \rangle = \frac{1}{K} \sum_{j=1}^{K} a_j = \frac{1}{K} \left(a_1 + a + 2 + a_3 + a_4 ... + a_K \right)$$

and similarly for b_j. If we now want the average of the sum of a and b we have

$$\langle a_j + b_j \rangle = \frac{1}{K} \sum_{j=1}^{K} (a_j + b_j) = \frac{1}{K} \left(a_1 + b_1 + a + 2 + b_2 + ... + a_K + b_K \right).$$

Grouping the right hand side we get

$$\langle a_j + b_j \rangle = \langle a_j \rangle + \langle a_b \rangle.$$

This means that the average of the sum is the sum of averages.

If we look at our model and consider

$$\langle C^n - H^n \rangle$$

we notice that apart from the minus sign C^n and H^n are generated the same way, so after lots of experiments both will have the same average. Thus

$$\langle C^n - H^n \rangle = 0$$

and thus

$$\langle P^{n+1} \rangle = \langle \lfloor P^n \rfloor \rangle$$

so that on average the population remains unchanged.

Question 5 If we have $\beta = 1$ and $\alpha = 0.8$ explain how you would find the number of conservationists needed to ensure that on average the population does not go down. Note: this is a difficult problem that you can consider as a challenge for after the session.

Solution:
Start with the equation

$$P^{n+1} = \lfloor (1.2)P^n \rfloor - H^n + C^n$$

and recall that the average is a additive, meaning we just apply it term by term and get

$$\langle P^{n+1} \rangle = \langle \lfloor (1.2)P^n \rfloor \rangle - \langle H^n \rangle + \langle C^n \rangle$$

We are told that the condition we want is $P^{n+1} \geq P^n$. so that

$$\langle \lfloor (1.2)P^n \rfloor \rangle - \langle H^n \rangle + \langle C^n \rangle \geq P^n.$$

Now just rearrange to have only the average of the conservationists on the left hand side and get

$$\langle C^n \rangle \geq P^n - \langle \lfloor (1.2)P^n \rfloor \rangle + \langle H^n \rangle.$$

That's the best we can do with algebra. The condition states that the number of conservationists we need depends on the number of hunters but also on the population at the time. Hence the conditions at the time and you can't give just one answer ahead of time. This bit of simple math makes you wonder about the simplistic answers you see to complicated problems on the news all the time!