Intermediate Math Circles
November 8, 2017
Probability II

Intersection of Events and Independence

Consider two groups of pairs of events

Group 1 (Dependent Events)

$A = \{$a sales associate has training$\}$
$B = \{$a sales associate is very helpful$\}$

(or if a pair of standard dice are rolled)
$A = \{6$ is rolled on first die$\}$
$B = \{6$ is rolled on both dice$\}$

Group 2 (Independent Events)

$A = \{$a person is right handed$\}$
$B = \{$a person is a brunette$\}$

(or if a pair of standard dice are rolled)
$A = \{5$ is rolled on first die$\}$
$B = \{5$ is rolled on second die$\}$

What do the pairs A and B in each group have in common?

- In Group 1 the events are related so the occurrence of A affects the chances of B occurring.
- In Group 2 whether A occurs or not has no effect on B's occurrence.

We can formalize this in a definition

Definition
Events A and B are independent if and only if

$$P(A \cap B) = P(A) \times P(B)$$

If they are not independent, we call the events dependant.

If 2 events are independent and have positive probabilities then their intersection must be non-empty. Therefore, independent events are normally not mutually exclusive.

Example 1:

Consider two events A and B where $P(A) = 0.6$, $P(B) = 0.35$ and $P(A \cap B) = 0.25$.
Are A and B independent?

Solution:

If A and B are independent, then $P(A \cap B) = P(A) \times P(B)$ must hold true. Since

$$P(A) \times P(B) = 0.6 \times 0.35 = 0.21 \neq 0.25$$

we conclude that A and B are not independent.

The definition of independence can be extended to a finite group of events. For example,

Definition
Events A, B, and C are mutually independent if and only if

$$P(A \cap B \cap C) = P(A) \times P(B) \times P(C)$$

$$\begin{aligned} P(A \cap B) &= P(A) \times P(B) \\ P(A \cap C) &= P(A) \times P(C) \\ P(B \cap C) &= P(B) \times P(C) \end{aligned}$$

Example 2:

Find the probability of rolling doubles three times in a row with a pair of dice.

Solution:

First, $P(\text{doubles}) = \dfrac{6}{36} = \dfrac{1}{6}$. Since each roll is independent of the others, we have

$$\begin{aligned} P(\text{doubles} \cap \text{doubles} \cap \text{doubles}) &= P(\text{doubles}) \times P(\text{doubles}) \times P(\text{doubles}) \\ &= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \\ &= \frac{1}{216} \end{aligned}$$

Example 3:

Three cards are drawn, with replacement, from an ordinary deck. Find the probability that exactly 2 of the cards are hearts.

Solution:

$$\begin{aligned} P(\text{exactly 2 hearts}) &= \binom{3}{2} P(\text{H H other}) \\ &= 3 \times \left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right) \\ &= \frac{9}{64} \end{aligned}$$

Example 4:

If the two events A and B are independent, prove that A^C and B are also independent.

Solution:

$$\begin{aligned}
P(A \cap B) &= P(A) \times P(B) \\
&= \left(1 - P(A^C)\right) \times P(B) \\
P(A \cap B) &= P(B) - P(A^C) \times P(B) \\
P(A^C) \times P(B) &= P(B) - P(A \cap B) \\
P(A^C) \times P(B) &= P(A^C \cap B)
\end{aligned}$$

Since $P(A^C) \times P(B) = P(A^C \cap B)$, by definition A^C and B are independent events.

Exercises

1. A die is rolled and a card is randomly drawn from a standard deck of 52 playing cards.
   (a) Determine the probability of rolling a prime number, then drawing a club.
   (b) Determine the probability of not rolling a 1, then drawing a face card.
2. Two weighted dice have $P(1) = P(3) = P(5) = 0.1$, $P(2) = P(4) = 0.2$ and $P(6) = 0.3$.
   (a) A pair of dice are rolled. What is the probability that the total is 8?
   (b) A pair of dice are rolled twice. What is the probability that the total will be 8 on both throws?
   (c) A pair of dice are rolled three times. What is the probability the total will be 8 on exactly 1 of the 3 throws?
3. Prove that A^C and B^C are independent if and only if A^C and B are independent by completing the following two steps.
   (a) Prove A^C and B are independent if you know that A^C and B^C are independent.
   (b) Prove A^C and B^C are independent if you know that A^C and B are independent.

Answers

1. (a) $\dfrac{1}{8}$
   (b) $\dfrac{5}{26}$
2. (a) $P(\text{sum} = 8) = 0.18$
   (b) $P(\text{sum} = 8 \text{ on both throws}) = 0.0324$
   (c) $P(\text{sum} = 8 \text{ on exactly 1 of 3 throws}) = 0.3631$

Conditional Probability

In many cases we want to determine the probability of some event A, while knowing that some other event B has already occurred.

Example 5:

Find the probability of drawing a 7 from a deck of cards and then drawing a jack on the next draw.

Solution:

Out of 52 cards 4 are 7's.

$$P(\text{draw }7) = \frac{4}{52} = \frac{1}{13}$$

Out of the 51 cards that are left, 4 are jacks.

$$P(\text{draw J after7}) = \frac{4}{51}$$

Therefore,

$$P(\text{draw }7) \times P(\text{draw J after }7) = \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$$

Notice that the probability of drawing a jack changes because it is dependent on the card that is chosen first.

Let $P(A|B)$ represents the probability that event A occurs when we know that B occurs. We call this the conditional probability of A given B.

Definition
The conditional probability of event A, given event B, is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \quad \text{provided } P(B) \neq 0$$

NOTE: If A and B are independent,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

This makes sense and we can use this as an equivalent definition of independence.
That is, A and B are independent if and only if $P(A|B) = P(A)$.

Example 6:

If a fair coin is tossed 3 times and you know that at least 1 head occurs. Find the probability that exactly 1 head occurs.

Solution:

$$\begin{aligned}
P(\text{exactly 1 H}|\text{at least 1 H}) &= \frac{P(\text{exactly 1 H} \cap \text{at least 1 H})}{P(\text{at least 1 H})} \\
&= \frac{\frac{3}{8}}{\frac{7}{8}} \\
&= \frac{3}{8} \div \frac{7}{8} \\
&= \frac{3}{7}
\end{aligned}$$

Example 7:

What is the probability that you roll a 12 with the throw of 2 dice, given that you know the second die is a 6.

Solution:

If the sum of the two dice is 12 and the second die is a 6, then we know that we are looking for the first die to also be a 6.

Since the two events are independent, we can use the alternative definition for independent events, that is $P(A|B) = P(A)$.

$$\begin{aligned}
P(\text{first} = 6|\text{second} = 6) &= P(\text{first} = 6) \\
&= \frac{1}{6}
\end{aligned}$$

Alternatively, if you did not notice that the two events were independent and worked through using the conditional probability formula, you would get the same answer.

$$\begin{aligned}
P(\text{first} = 6|\text{second} = 6) &= \frac{P(\text{first} = 6 \cap \text{second} = 6)}{P(\text{second} = 6)} \\
&= \frac{\frac{1}{36}}{\frac{1}{6}} \\
&= \frac{1}{36} \div \frac{1}{6} \\
&= \frac{1}{6}
\end{aligned}$$

Example 8:

The probability that a randomly selected male is colour-blind is 0.07 whereas the probability a female is colour-blind is only 0.0025. If the population is 45% male, what percent of the population is colour-blind?

Solution: Let

$$\begin{aligned} C &= \text{person selected is colour-blind} \\ M &= \text{person selected is male} \\ F &= \text{person selected is female} \end{aligned}$$

We are given that

$$P(C|M) = 0.07 \qquad P(C|F) = 0.0025 \qquad P(M) = 0.45 \qquad P(F) = 0.55$$

We want to find $P(C)$, which we can represent as

$$\begin{aligned} P(C) &= P(C \cap M) + P(C \cap F) \\ &= P(C|M)P(M) + P(C|F)P(F) \\ &= 0.07(0.45) + 0.0025(0.55) \\ &= 0.0315 + 0.001375 \\ &= 0.032875 \end{aligned}$$

Problem Set

1. (a) In a coin toss, what is the probability that heads is flipped exactly two times out of three tosses?

   Solution:

   There are $\binom{3}{2}$ ways to arrange 2 heads and 1 tail and each toss of the coin is independent from the other two. Therefore,

$$\begin{aligned} P(\text{exactly 2 heads}) &= \binom{3}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= 3\left(\frac{1}{8}\right) \\ &= \frac{3}{8} \\ &= 0.375 \end{aligned}$$

   (b) What if the coin is weighted so that the probability that heads occurs is 0.6?

   Solution:

   There are still $\binom{3}{2}$ ways to arrange 2 heads and 1 tail and each toss of the coin is independent from the other two. Therefore,

$$\begin{aligned} P(\text{exactly 2 heads}) &= \binom{3}{2}(0.6)(0.6)(0.4) \\ &= 3(0.36)(0.4) \\ &= 0.432 \end{aligned}$$

2. Three digits are chosen at random (with replacement) from $0, 1, \ldots, 9$. Find the probability of each of the following events.

   (a) A: "all three digits are the same"

   Solution:

   Each of the three draws is independent from the others. Therefore,

$$\begin{aligned} P(A) &= P(\text{any number}) \times P(\text{same number}) \times P(\text{same number}) \\ &= 1 \times \frac{1}{10} \times \frac{1}{10} \\ &= \frac{1}{100} \\ &= 0.01 \end{aligned}$$

(b) B: “all three digits are different”

Solution:

Each of the three draws is independent from the others. Therefore,

$$\begin{aligned}P(B) &= P(\text{any number}) \times P(\text{different number}) \times P(\text{another different number})\\ &= 1 \times \frac{9}{10} \times \frac{8}{10}\\ &= \frac{72}{100}\\ &= 0.72\end{aligned}$$

(c) C: “the digits all exceed 4”

Solution:

Each of the three draws is independent from the others. Therefore,

$$\begin{aligned}P(C) &= P(\text{number} > 4) \times P(\text{number} > 4) \times P(\text{number} > 4)\\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\\ &= \frac{1}{8}\\ &= 0.125\end{aligned}$$

(d) D: “digits are either all even or all odd”

Solution:

First we note that the digits being all even or all odd are mutually exclusive events. Within each event, each of the three draws is independent from the others. Therefore,

$$\begin{aligned}P(D) &= P(\text{all even}) + P(\text{all odd})\\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\\ &= \frac{1}{8} + \frac{1}{8}\\ &= \frac{1}{4}\\ &= 0.25\end{aligned}$$

3. A gumball machine contains 4 colours of gumballs. If you purchase a gumball, there is a 30% chance it will be blue, a 40% chance it will be green, a 10% chance it will be red, and a 20% chance it will be yellow. All of the red gumballs are removed. What are the chances of getting a blue, green and yellow gum ball now?

   Solution:

   We want to determine $P(B|R^C)$, $P(G|R^C)$ and $P(Y|R^C)$.

$$P(B|R^C) = \frac{P(B \cap R^C)}{P(R^C)} = \frac{0.3}{0.9} = \frac{1}{3} \qquad P(G|R^C) = \frac{P(G \cap R^C)}{P(R^C)} = \frac{0.4}{0.9} = \frac{4}{9} \qquad P(Y|R^C) = \frac{P(Y \cap R^C)}{P(R^C)} = \frac{0.2}{0.9} = \frac{2}{9}$$

   Therefore, the chance of getting blue is $\frac{1}{3}$, the chance of getting a green is $\frac{4}{9}$, and the chance of getting a yellow is $\frac{2}{9}$.

4. In Canada, the probability that someone plays baseball or hockey is 0.79. The probability that someone plays just hockey is 0.6 and the probability that someone plays baseball and hockey is 0.15. What is the probability that someone plays only baseball?

   Solution:

   We are given that $P(B \cup H) = 0.79$, $P(H) = 0.6$ and $P(B \cap H) = 0.15$.

   We also know that $P(B \cup H) = P(B) + P(H) - P(B \cap H)$ which we can rearrange to solve for $P(B)$.

$$\begin{aligned} P(B) &= P(B \cup H) + P(B \cap H) - P(H) \\ &= 0.79 + 0.15 - 0.6 \\ &= 0.34 \end{aligned}$$

   So we have that $P(B) = 0.34$. But this is not our final answer, because we want to find the probability that someone plays only baseball. That means we want to find $P(B \cap H^C)$ (you can convince yourself of this by drawing a venn diagram).

$$\begin{aligned} P(B \cap H^C) &= P(B) - P(B \cap H) \\ &= 0.34 - 0.15 \\ &= 0.14 \end{aligned}$$

   Therefore, the probability that a student plays only baseball is 0.14.

5. Let A and B be events defined on the same sample space, with $P(A) = 0.3$, $P(B) = 0.4$ and $P(A|B) = 0.5$. If you are given that B does **not** occur, what is the probability of event A occurring?

   Solution:

   We want to find $P(A|B^C)$.

   We know that $P(A|B) = 0.5$. We can use the formula for conditional probability and show that

$$\begin{aligned} P(A \cap B) &= P(A|B) \times P(B) \\ &= 0.5 \times 0.4 \\ &= 0.2 \end{aligned}$$

   Notice that $P(A) \times P(B) \neq P(A \cap B)$ and so the events A and B are not independent.

   Next, you can draw a venn diagram to convince yourself that

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^C) \\ P(A \cap B^C) &= P(A) - p(A \cap B) \\ &= 0.3 - 0.2 \\ &= 0.1 \end{aligned}$$

   We can now use the formula for conditional probability and we have

$$\begin{aligned} P(A|B^C) &= \frac{P(A \cap B^C)}{P(B^C)} \\ &= \frac{0.1}{0.6} \\ &= \frac{1}{6} \end{aligned}$$

   Therefore, if you are given that event B does not occur, then the probability of event A occurring is $\frac{1}{6}$.

6. Billy and Crystal each have a bag of 9 balls. The balls in each bag are numbered 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the balls remaining in Billy's bag. Let c be the sum of the numbers on the balls remaining in Crystal's bag.
Determine the probability that b and c differ by a multiple of 4.

Solution:

Suppose that Billy removes the ball numbered x from his bag and Crystal removes the ball numbered y from her bag.
Then $b = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 - x = 45 - x$.
Also $c = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 - y = 45 - y$.
Hence, $b - c = (45 - x) - (45 - y) = y - x$.

Since $1 \leq x \leq 9$ and $1 \leq y \leq 9$, we have that $-8 \leq y - x \leq 8$.
(This is because $y - x$ is maximized when y is largest (that is $y = 9$) and x is smallest (that is when $x = 1$), so $y - x \leq 9 - 1 = 8$. A similar argument can be made for when $y - x \geq -8$).

Since $b - c = y - x$ is between -8 and 8, in order for it to be a multiple of 4, $b - c = y - x$ can be $-8, -4, 0, 4$, or 8.
Since each of Billy and Crystal choose 1 ball from 9 balls and each ball is equally likely to be chosen, the probability of any specific ball being chosen from one if their bags is $\frac{1}{9}$. Thus, the probability of any specific pair of balls being chosen (one from each bag) is $\frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$.
To compute the desired probability, we must count the number of pairs (x, y) where $y - x$ is $-8, -4, 0, 4$, or 8 and then multiply the result by $\frac{1}{81}$.
If $y - x = -8$, then (x, y) must be $(9, 1)$.
If $y - x = 8$, then (x, y) must be $(1, 9)$.
If $y - x = -4$, then (x, y) can be $(5, 1), (6, 2), (7, 3), (8, 4), (9, 5)$.
If $y - x = 4$, then (x, y) can be $(1, 5), (2, 6), (3, 7), (4, 8), (5, 9)$.
If $y - x = 0$, then (x, y) can be $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)$.

Therefore, there are 21 pairs (x, y) that work, so the desired probability is $\frac{21}{81} = \frac{7}{27}$.