Intermediate Math Circles
November 8, 2017
Probability II

Problem Set Solutions

1. (a) In a coin toss, what is the probability that heads is flipped exactly two times out of three tosses?

   Solution:

   There are $\binom{3}{2}$ ways to arrange 2 heads and 1 tail and each toss of the coin is independent from the other two. Therefore,

$$\begin{aligned} P(\text{exactly 2 heads}) &= \binom{3}{2}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= 3\left(\frac{1}{8}\right) \\ &= \frac{3}{8} \\ &= 0.375 \end{aligned}$$

   (b) What if the coin is weighted so that the probability that heads occurs is 0.6?

   Solution:

   There are still $\binom{3}{2}$ ways to arrange 2 heads and 1 tail and each toss of the coin is independent from the other two. Therefore,

$$\begin{aligned} P(\text{exactly 2 heads}) &= \binom{3}{2}(0.6)(0.6)(0.4) \\ &= 3(0.36)(0.4) \\ &= 0.432 \end{aligned}$$

2. Three digits are chosen at random (with replacement) from $0, 1, \ldots, 9$. Find the probability of each of the following events.

   (a) A: "all three digits are the same"

   Solution:

   Each of the three draws is independent from the others. Therefore,

$$\begin{aligned} P(A) &= P(\text{any number}) \times P(\text{same number}) \times P(\text{same number}) \\ &= 1 \times \frac{1}{10} \times \frac{1}{10} \\ &= \frac{1}{100} \\ &= 0.01 \end{aligned}$$

(b) B: "all three digits are different"

Solution:

Each of the three draws is independent from the others. Therefore,

$$\begin{aligned} P(B) &= P(\text{any number}) \times P(\text{different number}) \times P(\text{another different number}) \\ &= 1 \times \frac{9}{10} \times \frac{8}{10} \\ &= \frac{72}{100} \\ &= 0.72 \end{aligned}$$

(c) C: "the digits all exceed 4"

Solution:

Each of the three draws is independent from the others. Therefore,

$$\begin{aligned} P(C) &= P(\text{number} > 4) \times P(\text{number} > 4) \times P(\text{number} > 4) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \frac{1}{8} \\ &= 0.125 \end{aligned}$$

(d) D: "digits are either all even or all odd"

Solution:

First we note that the digits being all even or all odd are mutually exclusive events. Within each event, each of the three draws is independent from the others. Therefore,

$$\begin{aligned} P(D) &= P(\text{all even}) + P(\text{all odd}) \\ &= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \\ &= \frac{1}{8} + \frac{1}{8} \\ &= \frac{1}{4} \\ &= 0.25 \end{aligned}$$

3. A gumball machine contains 4 colours of gumballs. If you purchase a gumball, there is a 30% chance it will be blue, a 40% chance it will be green, a 10% chance it will be red, and a 20% chance it will be yellow. All of the red gumballs are removed. What are the chances of getting a blue, green and yellow gum ball now?

   Solution:

   We want to determine $P(B|R^C)$, $P(G|R^C)$ and $P(Y|R^C)$.

$$P(B|R^C) = \frac{P(B \cap R^C)}{P(R^C)} = \frac{0.3}{0.9} = \frac{1}{3} \qquad P(G|R^C) = \frac{P(G \cap R^C)}{P(R^C)} = \frac{0.4}{0.9} = \frac{4}{9} \qquad P(Y|R^C) = \frac{P(Y \cap R^C)}{P(R^C)} = \frac{0.2}{0.9} = \frac{2}{9}$$

   Therefore, the chance of getting blue is $\frac{1}{3}$, the chance of getting a green is $\frac{4}{9}$, and the chance of getting a yellow is $\frac{2}{9}$.

4. In Canada, the probability that someone plays baseball or hockey is 0.79. The probability that someone plays just hockey is 0.6 and the probability that someone plays baseball and hockey is 0.15. What is the probability that someone plays only baseball?

   Solution:

   We are given that $P(B \cup H) = 0.79$, $P(H) = 0.6$ and $P(B \cap H) = 0.15$.

   We also know that $P(B \cup H) = P(B) + P(H) - P(B \cap H)$ which we can rearrange to solve for $P(B)$.

$$\begin{aligned} P(B) &= P(B \cup H) + P(B \cap H) - P(H) \\ &= 0.79 + 0.15 - 0.6 \\ &= 0.34 \end{aligned}$$

   So we have that $P(B) = 0.34$. But this is not our final answer, because we want to find the probability that someone plays only baseball. That means we want to find $P(B \cap H^C)$ (you can convince yourself of this by drawing a venn diagram).

$$\begin{aligned} P(B \cap H^C) &= P(B) - P(B \cap H) \\ &= 0.34 - 0.15 \\ &= 0.14 \end{aligned}$$

   Therefore, the probability that a student plays only baseball is 0.14.

5. Let A and B be events defined on the same sample space, with $P(A) = 0.3$, $P(B) = 0.4$ and $P(A|B) = 0.5$. If you are given that B does **not** occur, what is the probability of event A occurring?

Solution:

We want to find $P(A|B^C)$.

We know that $P(A|B) = 0.5$. We can use the formula for conditional probability and show that

$$\begin{aligned} P(A \cap B) &= P(A|B) \times P(B) \\ &= 0.5 \times 0.4 \\ &= 0.2 \end{aligned}$$

Notice that $P(A) \times P(B) \neq P(A \cap B)$ and so the events A and B are not independent.

Next, you can draw a venn diagram to convince yourself that

$$\begin{aligned} P(A) &= P(A \cap B) + P(A \cap B^C) \\ P(A \cap B^C) &= P(A) - p(A \cap B) \\ &= 0.3 - 0.2 \\ &= 0.1 \end{aligned}$$

We can now use the formula for conditional probability and we have

$$\begin{aligned} P(A|B^C) &= \frac{P(A \cap B^C)}{P(B^C)} \\ &= \frac{0.1}{0.6} \\ &= \frac{1}{6} \end{aligned}$$

Therefore, if you are given that event B does not occur, then the probability of event A occurring is $\frac{1}{6}$.

6. Billy and Crystal each have a bag of 9 balls. The balls in each bag are numbered 1 to 9. Billy and Crystal each remove one ball from their own bag. Let b be the sum of the numbers on the balls remaining in Billy's bag. Let c be the sum of the numbers on the balls remaining in Crystal's bag.
Determine the probability that b and c differ by a multiple of 4.

<u>Solution:</u>

Suppose that Billy removes the ball numbered x from his bag and Crystal removes the ball numbered y from her bag.
Then $b = 1+2+3+4+5+6+7+8+9-x = 45-x$.
Also $c = 1+2+3+4+5+6+7+8+9-y = 45-y$.
Hence, $b-c = (45-x)-(45-y) = y-x$.

Since $1 \le x \le 9$ and $1 \le y \le 9$, we have that $-8 \le y-x \le 8$.
(This is because $y-x$ is maximized when y is largest (that is $y = 9$) and x is smallest (that is when $x = 1$), so $y-x \le 9-1 = 8$. A similar argument can be made for when $y-x \ge -8$).

Since $b-c = y-x$ is between -8 and 8, in order for it to be a multiple of 4, $b-c = y-x$ can be $-8, -4, 0, 4$, or 8.
Since each of Billy and Crystal choose 1 ball from 9 balls and each ball is equally likely to be chosen, the probability of any specific ball being chosen from one if their bags is $\frac{1}{9}$. Thus, the probability of any specific pair of balls being chosen (one from each bag) is $\frac{1}{9} \times \frac{1}{9} = \frac{1}{81}$.
To compute the desired probability, we must count the number of pairs (x, y) where $y-x$ is $-8, -4, 0, 4$, or 8 and then multiply the result by $\frac{1}{81}$.
If $y-x = -8$, then (x, y) must be $(9, 1)$.
If $y-x = 8$, then (x, y) must be $(1, 9)$.
If $y-x = -4$, then (x, y) can be $(5, 1), (6, 2), (7, 3), (8, 4), (9, 5)$.
If $y-x = 4$, then (x, y) can be $(1, 5), (2, 6), (3, 7), (4, 8), (5, 9)$.
If $y-x = 0$, then (x, y) can be $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 7), (8, 8), (9, 9)$.

Therefore, there are 21 pairs (x, y) that work, so the desired probability is $\frac{21}{81} = \frac{7}{27}$.