



Grade 6 Math Circles
7/8 November, 2017
Approximations & Estimations

Solutions

Example 1

- (a) The question asks you to estimate the decimal value of $\frac{5}{12}$. To do this, we first need known fractions that can work as our upper and lower bounds.

For an upper bound, notice that $\frac{5}{12}$ is smaller than $\frac{6}{12}$. We already know that $\frac{6}{12} = \frac{1}{2} = 0.5$. This means that our upper bound is $\frac{6}{12}$.

We use $\frac{6}{12}$ for our upper bound instead of the equivalent $\frac{1}{2}$ because it's easy to see why $\frac{6}{12}$ is greater than our given fraction. This is a very helpful thing to do - always try and rewrite your bounds so that you can compare them easily to the fraction you need to estimate.

For our lower bound, notice that $\frac{4}{12}$ is smaller than $\frac{5}{12}$. We also know that $\frac{4}{12} = \frac{1}{3} = 0.33$. This means that our lower bound will be $\frac{4}{12}$.

So we can write our fraction and bounds as:

$$\frac{4}{12} < \frac{5}{12} < \frac{6}{12}$$

Remember that since we're doing an estimation problem, we could use any bounds we like. For example, we could've used the bounds 0 and 100. But these bounds aren't very useful because there's a big range of possible answers. For good estimates, we want our bounds to be as close to the fraction we're estimating.

Using the known decimal values for our bounds, it must mean that:

$$0.33 < \frac{5}{12} < 0.50$$

In other words, the decimal value of $\frac{5}{12}$ must lie in between 0.33 and 0.50. We've managed to narrow down our answer to a very small interval. Now all we need to do is guess a number in between our bounds.

Suppose I guess a value of 0.4 (since it's in the middle). This means that:

$$\frac{5}{12} \approx 0.4$$

If you try finding the decimal value of $\frac{5}{12}$ on a calculator, you'd find that it's actually 0.4166.. repeating. So we were really close with our estimate!

It's important to remember than when estimating, there's no real right answer - a good estimate is an answer that's **least wrong but still wrong**.

(b) Similar to part(a), we find upper and lower bounds to estimate this fraction.

For our lower bound, notice that $\frac{7}{14}$ is smaller than $\frac{7}{13}$. This is because $\frac{7}{14}$ has a bigger denominator. We also know the decimal value of $\frac{7}{14}$ - it's just 0.5. This means that our lower bound is $\frac{7}{14}$.

Similarly, for our upper bound, we know that $\frac{8}{12}$ must be greater than $\frac{7}{13}$. We also know that $\frac{8}{12} = \frac{2}{3} = 0.66\dots$. This means that $\frac{8}{12}$ is a good upper bound for us to use.

Now we can write our fraction in between the upper and lower bounds:

$$\frac{7}{14} < \frac{7}{13} < \frac{8}{12}$$

Remember that the way we come up with upper and lower bounds is by making clever guesses. An easy way to think of it, is to think of changing the numerator or denominator of our fraction to be estimated, up or down by 1 or 2. For example, changing the denominator of $\frac{7}{13}$ down by 1 gives $\frac{7}{14}$. The only way to get good at finding bounds, is to do many estimation problems on your own. Try making up your own!

Now, using the known decimal values of our upper and lower bounds, we get:

$$0.50 < \frac{7}{13} < 0.66$$

So we know our estimate must be in between 0.50 and 0.66. Again, we guess a number in that interval - say 0.58.

So:

$$\frac{7}{13} \approx 0.58$$

It turns out that $\frac{7}{13}$ is actually 0.538. So our estimate wasn't too far off! You might've gotten a slightly different estimate, but that's completely alright as long as it is within 0.50 and 0.66.

- (c) Our upper bound for this question is pretty easy to get - we change the denominator down by 1 to get $\frac{9}{18}$ which we know is the same as 0.5.

Our lower bound is a bit trickier. Notice that if we change our denominator up by 2, we can get $\frac{9}{21}$ which simplifies to $\frac{3}{7}$. The problem is that we don't know the decimal value of $\frac{3}{7}$.

Now we've got two choices - either find a new lower bound with a decimal value that we already know, or estimate $\frac{3}{7}$.

For the purposes of this question, we're going to find a new lower bound instead of estimating $\frac{3}{7}$ (although it's perfectly right to do so).

Notice that you can get the fraction $\frac{6}{18}$ (which is known to be the same as 0.33) by changing the numerator down by 3 and the denominator down by 1. We know that $\frac{9}{19}$ must be greater than 0.33 because $9 \times 3 = 27$ i.e. $\frac{9}{27} = 0.33$. Our fraction has the same numerator but a smaller denominator, which means that our fraction must be bigger than 0.33.

So writing our fraction within its bounds, we get:

$$\frac{6}{18} < \frac{9}{19} < \frac{9}{18}$$

This means that:

$$0.33 < \frac{9}{19} < 0.5$$

Making a guess, we get 0.40. So we write:

$$\frac{9}{19} \approx 0.40$$

It turns out that the actual decimal value of $\frac{9}{19}$ is 0.47. Again, we were really close with our estimate.

(d) We need to find an estimate for $\frac{17}{9}$. This is slightly different from the previous questions because this fraction is **improper** i.e. the numerator is bigger than the denominator. However, this doesn't really change our method of estimation - we still need to find upper and lower bounds.

For our upper bound, notice that increasing the numerator by 1 gives $\frac{18}{9}$, which is just 2. This is a good upper bound to have.

For our lower bound, we notice that decreasing the numerator by 2 gives us $\frac{15}{9}$. We know that $\frac{15}{9} = \frac{5}{3}$. Now, as in part(c), we've got two choices - find a new lower bound or get an estimate for $\frac{5}{3}$. This time, we're going to find an estimate for $\frac{5}{3}$.

We can rewrite $\frac{5}{3}$ as:

$$\frac{5}{3} = \frac{2}{3} + \frac{3}{3} = \frac{2}{3} + 1$$

We already know that $\frac{2}{3} = 0.66$. This means that:

$$\frac{5}{3} = \frac{2}{3} + 1 = 1 + 0.66 = 1.66$$

Writing our fraction within its bounds we get:

$$\frac{15}{9} < \frac{17}{9} < \frac{18}{9}$$

This means that:

$$1.66 < \frac{17}{9} < 2$$

We guess a number between 1.66 and 2 - say 1.7. So we write:

$$\frac{17}{9} \approx 1.7$$

It turns out that the actual decimal value for $\frac{17}{9}$, is 1.888... repeating. Remember to not worry if you didn't get the exact answer as given here. As long as you're close, you're right.

Example 2

- (a) (i) The question asks you to find a decimal approximation of $\sqrt{43}$ to two decimal places.

To start, we do the same thing we did with fractions - we find upper and lower bounds.

For our bounds, notice that $6^2 = 6 \times 6 = 36$. Also, $7^2 = 7 \times 7 = 49$.

This means that $\sqrt{43}$ must be in between $\sqrt{36}$ and $\sqrt{49}$. In other words:

$$\sqrt{36} < \sqrt{43} < \sqrt{49}$$

This means that $\sqrt{43}$ must be in between 6 and 7:

$$6 < \sqrt{43} < 7$$

This is where we start to deviate from the method we used for fractions. We start by making our guess a number exactly in the middle of our interval i.e. we guess 6.50. Let's call this our first guess.

We quickly check the value of 6.5^2 which is 42.25. This is a bit too low - we need 43.

So, we make a new guess (our 2^{nd} guess) by increasing our first guess by 0.1. Our second guess is now 6.6.

We see that $6.6^2 = 43.56$ is a bit too big. At this stage, we say that our first guess was closer and try and find the second decimal place.

So our guess was 6.50. We know that this is slightly too small. We also know that 6.6 is slightly too big. So we increase 6.50 by smaller 0.01 increments.

First we try 6.51, whose square gives $6.51^2 = 42.38$. That's still too low. We do the same for 6.52, 6.53 and 6.54 (all of which are slightly too low). Finally we land on 6.56 whose square is $6.56^2 = 43.03$ which is really close to 43.

Since 6.56^2 is so close to 43, we're satisfied and say that:

$$\sqrt{43} \approx 6.56 \text{ to 2 decimal places}$$

The actual answer for $\sqrt{43}$ is 6.557. Again, notice how close we got to the actual answer by making clever guesses.

- (ii) Similar to the previous problem, we start by finding our upper and lower bounds. Notice that $\sqrt{81} = 9$ and $\sqrt{64} = 8$. This means that:

$$\sqrt{64} < \sqrt{77} < \sqrt{99}$$

In other words:

$$8 < \sqrt{77} < 9$$

So our answer must be a number in between 8 and 9. We start with a guess that's right in the middle i.e. we guess 8.5.

$8.5^2 = 72.25$ which is too low. Our next guess would normally be 8.6 but since 8.5^2 was pretty far off from 77, we expect that changing our guess by 0.1 doesn't really make much of a difference. Instead, we change our guess by a larger step of 0.2 to 8.7.

We see that $8.7^2 = 75.69$. Our new guess is also a bit too low. Just to check, we could do find the square of 8.8 which is $8.8^2 = 77.44$. Since 8.8^2 is too high, we settle for 8.7 as our guess.

Now we need to find the answer to two decimal places. We start with a guess of 8.71 whose square gives $8.71^2 = 75.86$. This is still too low. We do the same thing for 8.72, 8.73 and so on, until we get to 8.78.

$8.78^2 = 77.08$ is really close to 77. To check, we could go a bit further and find $8.76^2 = 76.73$.

So 8.7^2 is closer to 77 than 8.76^2 . This means that we can write:

$$\sqrt{77} \approx 8.78$$

$\sqrt{77}$ is actually 8.7749. Our approximation was within 0.01 of the right answer!

- (b) This is an example of a question that requires you to use the Pythagorean theorem.

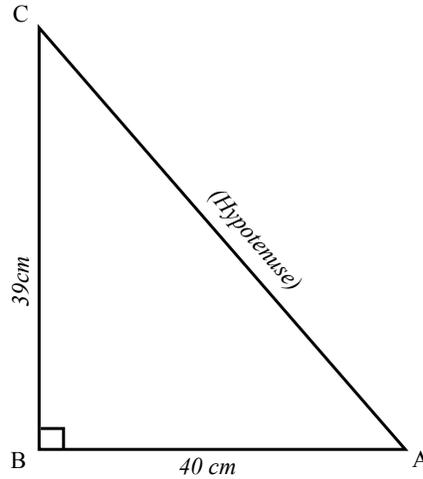


Figure 1: Example 2(b)

The Pythagorean theorem tells you that:

$$(base)^2 + (height)^2 = (hypotenuse)^2$$

In other words, it tells you that the hypotenuse is:

$$\sqrt{(base)^2 + (height)^2} = hypotenuse$$

The base of our triangle is 40 *cm* and the height of our triangle is 39 *cm*. Plugging these numbers into our formula, we get:

$$\sqrt{(40)^2 + (39)^2} = hypotenuse$$

This simplifies to:

$$\sqrt{1600 + 1521} = \sqrt{3121} = hypotenuse$$

Now all we've got to do is find an approximation of $\sqrt{3121}$ to one decimal place. Notice that this approximation is a bit different from the ones we did earlier, in that we're asked to approximate the square root of a 4 digit number.

We only know the square numbers from 1 to 10 - how're we going to do this? There's a pretty neat trick we could use. Notice that $6^2 = 36$. If we multiplied each 6 by a 10, we'd see that $60^2 = 3600$.

Similarly, $5^2 = 25$. Multiplying each 5 by a 10 gives $50^2 = 2500$.

So we've found our upper and lower bounds by being clever with how we used our table of square numbers. This means that we can write:

$$\sqrt{2500} < \sqrt{3121} < \sqrt{3600}$$

So our answer must be in between 50 and 60. We start with a guess that's in the middle of this interval - 55.

$55^2 = 3025$ which is slightly less than our answer. Also, $56^2 = 3136$ which is too high. So we let 55 be our first guess. Now all we need is the first decimal place.

Since we know that 56 is too big, we make a guess of 55.5. $55.5^2 = 3080.25$ which is still a bit too low. Doing exactly what we did earlier, we increase our guess to 55.6 and then later to 55.7. If you try and find the squares of these numbers, you'll see that they're both still too low.

Finally we get to 55.8. $55.8^2 = 3113.64$. It's still a bit too low. But $55.9^2 = 3124.81$ is too high.

So we say that 55.8 is our approximation for $\sqrt{3121}$. In other words:

$$\sqrt{3121} \approx 55.8$$

So the length of our hypotenuse is approximately 55.8 *cm*.

Example 3

(a) 4×10^0

Note: Remember that any number raised to 0 is just 1.

(b) 9.5×10^0

(c) 5.5×10^1

(d) 1.77×10^2

(e) 1.390×10^3

(f) 1.5092700×10^7

Example 4:

This is a pretty interesting problem. It requires you to estimate the number of piano tuners in the city of Chicago. A piano tuner is a person who comes to your house and tunes your (acoustic) piano in a couple of hours.

To solve this problem, we're going to need to make some assumptions (which may not necessarily be 100% true). Remember that the solution shown here is just one possible way of getting an estimate - you could try your own!

First, we're going to assume that every piano needs to be tuned at least once a year. The reason for this is that we plan to get an estimate for the number of pianos to be tuned in Chicago in a year. We also plan to get an estimate for the number of pianos a typical piano tuner tunes in a year. If we divide these two estimates, we should get an estimate for the number of piano tuners in Chicago.

To estimate the number of pianos to be tuned in a year, we first assume that a household is more likely to have a piano than an individual. Given that the population of Chicago is 9,000,000, this means that there must be 4,500,000 households (if we assume that two people make up a typical household). It's important to realize that this estimate need not be perfectly accurate - that's the point of an estimation. For example, you'll see that if we assumed that there were 4 people in a typical household (instead of 2) our final answer wouldn't change by too much.

So we have an estimate of the number of households in Chicago. Next, we need to find how many of these households have a piano (we're assuming that a household has a maximum of 1 piano). Notice that it's very unlikely that every single household in Chicago has a piano. It's also unlikely that every other household in Chicago has a piano. This means our estimate must be somewhere in between. Let's suppose that 1 in every 20 households have a piano - it seems reasonable.

This means that there must be $\frac{4,500,000}{20} = 225,000$ pianos in Chicago. Since we assumed that each piano must be tuned at least once in a year, this means that there are around 225,000 pianos in Chicago that require tuning each year.

Now for the second part of our estimate we need to find the number of pianos tuned by a

single tuner in a year. To do this, first we notice that a typical person works 8 hours a day. So in a week, a piano tuner works for $5 \text{ days} \times 8 \text{ hours} = 40 \text{ hours}$. We know that each month has about 4 weeks and that there are 12 months in a year. This means that there's about 50 weeks in a year (the actual number is 52 but remember that this is an estimate - it's alright to be off by a bit).

So a typical piano tuner works for $40 \text{ hours} \times 50 = 2000 \text{ hours}$ a year. Now, let's say that tuning a piano takes about 2 hours on average. This means that the number of pianos tuned by a single piano tuner in one year is about $\frac{2000 \text{ hours}}{2 \text{ hours per piano}} = 1000 \text{ pianos}$.

Now that we've got our two estimates, we can put them together and estimate the number of piano tuners in Chicago:

$$\# \text{ of Piano Tuners in Chicago} = \frac{\# \text{ of pianos to be tuned in a year in Chicago}}{\# \text{ of pianos tuned by a single piano tuner in a year}} = \frac{225,000}{1000} = 225$$

So according to our estimate, there are about 225 piano tuners in Chicago. It turns out that the actual answer (as of 2009) is 290. So we weren't really far off with our estimate!

It's important to remember that when you do this estimation on your own, you might end up with a different answer. This is completely alright. As long as your answer is still a 3-digit number, you're considered close enough. It's the method that really counts in an estimation problem.

Example 5

This problem asks you to estimate the number of hairs on an average human head. Well one way to solve it would be if we knew the surface area of an average head and the number of hairs per unit area. In other words:

$$\# \text{ of hairs} = (\text{Surface area of average human head}) \times (\# \text{ of hairs per unit area})$$

There are many ways to estimate the surface area of an average head. One way to do it would be to notice that a small-medium sized towel usually wraps around most people's heads completely. So we can assume that the area of such a towel is approximately the area of a head. A typical towel is about $30 \text{ cm} \times 30 \text{ cm}$. This means that the area of the towel is:

$$\text{Area of towel} = 30 \times 30 = 900 \text{ sq.cm}$$

Now if we go up to a mirror and pinch our hair we usually end up with about 15- 30 hairs. Let's say we get 15 (you could do 30, it wouldn't change the answer by much). A typical pinch is about a centimeter which means that the number of hairs per *sq. cm* must be:

$$\# \text{ of hairs per unit area} = 15 \times 15 = 225 \text{ hairs per sq.cm}$$

Now that we've got our two estimates, time to put them together.

$$\begin{aligned} \# \text{ of hairs} &= (\text{Surface area of head}) \times (\# \text{ of hairs per unit area}) = \\ &900 \text{ sq.cm} \times 225 \text{ hairs per sq.cm} = 202,500 \text{ hairs} \end{aligned}$$

A quick Google search tells us that the average number of hairs is between 100,000 and 150,000. Again, our estimate wasn't *correct* but it was close enough.

The method shown above is just one way of solving this problem. Try finding your own!

Example 6:

To do this estimate, we need two things - the approximate number of people in the room and the average height of a person in this room. The first one is easy, you can simply count the number of people. Let's say there are 30 people in the room (your estimate might be a bit higher or lower).

The second one is a bit harder to find. To find the average height, we can use a neat trick - we set upper and lower bounds, just like we did for square roots and fractions. Notice that 100 *cm* is too short to be the average height of the class. On the other hand, 200 *cm* is clearly too tall. So let's say our average height is exactly in the middle at 150 *cm*.

Now that we've got both our estimates, let's find the answer to our problem:

$$\begin{aligned} \text{Length of line} &= (\# \text{ of people in the room}) \times (\text{Average height of class}) = 30 \times 150 \text{ cm} = \\ &4500 \text{ cm} \end{aligned}$$

We don't immediately know how much 4500 *cm* is so we convert it to meters by dividing our answer by 100:

$$4500 \text{ cm} = 45 \text{ m}$$

45 m is actually pretty long - in fact, it's wider than the room! In this case we don't have an actual answer for us to compare against - we'd actually have to try it out to get an answer. But the estimate gives us a rough idea of how big the line might be if we tried the experiment.

Problem Set

1. (a) First, we need upper and lower bounds.

$$\frac{6}{8} < \frac{7}{8} < \frac{8}{8}$$

This means that:

$$0.75 < \frac{7}{8} < 1$$

Now we guess a number between 0.75 and 1. Let's say our guess is 0.8. So we write:

$$\frac{7}{8} \approx 0.8$$

The actual answer is 0.87 which is really close to our estimate.

- (b) Again, we need upper and lower bounds.

$$\frac{21}{7} < \frac{22}{7} < \frac{28}{7}$$

This means that:

$$3 < \frac{22}{7} < 4$$

So we need to guess a number between 3 and 4. Let's say we guess 3.5. So we write:

$$\frac{22}{7} \approx 3.5$$

The actual answer is 3.14 (or π), as you may already know.

- (c) This problem can be done in two ways - you could either ignore the mixed fraction or convert it to an improper fraction and follow the method we used in part (b). The first method is shown here because it's slightly different to what we've done until now.

First, we ignore the 5 in the mixed fraction. So now we need to estimate $\frac{2}{5}$. As before, we need upper and lower bounds.

$$\frac{2}{6} < \frac{2}{5} < \frac{2}{4}$$

This means that $\frac{2}{5}$ must be in between:

$$0.33 < \frac{2}{5} < 0.5$$

We guess that the answer is 0.4. So we can write:

$$\frac{2}{5} \approx 0.4$$

So now that we've estimated 0.4, we simply add 5 to it to estimate $5\frac{2}{5}$. In other words:

$$5\frac{2}{5} \approx 5.4$$

This time, our estimate was exact - the actual answer is also 5.4! Notice that if our guess for the decimal value of $\frac{2}{5}$ was different, our final estimate would also be different.

(d) First, we get our upper and lower bounds:

$$\frac{28}{14} < \frac{29}{11} < \frac{30}{10}$$

This means that:

$$2 < \frac{29}{11} < 3$$

We guess that the decimal value is 2.5. So we write:

$$\frac{29}{11} \approx 2.5$$

The actual value is 2.63. Again, our estimate is really close to the actual answer.

(e) This question is different because you need to do two estimations to get the answer. First, we'll estimate the square root in the numerator. Then we'll estimate the fraction.

To estimate $\sqrt{41}$ we need upper and lower bounds - refer to the square roots table in the handout if you need to.

$$\sqrt{36} < \sqrt{41} < \sqrt{49}$$

This means that:

$$6 < \sqrt{41} < 7$$

For our first guess, we pick 6.5. Checking the square we see that $6.5^2 = 42.25$ which is a bit too high.

For our next guess, we pick 6.4, whose square gives $6.4^2 = 40.96$ which is slightly too low, but closer to 41 than 6.5^2 .

To find our second decimal place, we notice that our answer must be in between 6.4 and 6.5 i.e.:

$$6.4 < \sqrt{41} < 6.5$$

We start with a guess that's slightly higher than 6.4 (because 6.4^2 was very close to 41). We start with 6.41. Checking the square, this gives $6.41^2 = 41.08$. This is slightly above our required answer so we can pick either 6.40 or 6.41 as our final estimate - we pick 6.40 here.

So our fraction is approximately $\frac{6.4}{12}$. To estimate this, we need upper and lower bounds:

$$\frac{6}{12} < \frac{6.4}{12} < \frac{8}{12}$$

In other words:

$$0.5 < \frac{6.4}{12} < 0.66$$

Now all we need to do is to guess a number in the above interval - say 0.60. So:

$$\frac{\sqrt{41}}{12} \approx 0.60$$

The actual answer in this case is 0.53.

2. (a) As usual, we find upper and lower bounds first.

$$\sqrt{25} < \sqrt{30} < \sqrt{36}$$

This means that:

$$5 < \sqrt{30} < 6$$

For our first guess we pick 5.5, whose square gives $5.5^2 = 30.25$. Since this is a bit too high, we go lower and make 5.4 our next guess. $5.4^2 = 29.16$ is slightly too low. We now know that our answer lies in between 5.4 and 5.5.

$$5.4 < \sqrt{30} < 5.5$$

We start with a guess slightly less than 5.5 because 5.5^2 was closer to 30 than 5.4^2 . Let's say our guess is 5.48. The square is $5.48^2 = 30.03$. This is really close to 30. So we say:

$$\sqrt{30} \approx 5.48$$

The actual answer in this case is 5.477.

- (b) We write our upper and lower bounds again:

$$\sqrt{64} < \sqrt{71} < \sqrt{81}$$

In other words:

$$8 < \sqrt{719}$$

Our first guess is 8.5. Checking the square we find that $8.5^2 = 72.25$. Since this is too high, we go lower and check 8.4. $8.4^2 = 70.56$. This is a bit too low but we can use this as a first guess.

Now we know that our answer must lie in between 8.5 and 8.4.

$$8.4 < \sqrt{71} < 8.5$$

Since 8.4^2 was really close to 71 we go up by a bit and guess 8.41. Checking the square we see that $8.41^2 = 70.72$, which is still a tiny bit too low. Changing our guess by 0.01 we try 8.42, which gives $8.42^2 = 70.89$. This is also too low. If we keep going we find that our estimate gets close to 8.43. So we write:

$$\sqrt{71} \approx 8.43$$

The actual answer in this case is 8.426.

(c) Our upper and lower bounds in this case are:

$$\sqrt{100} < \sqrt{120} < \sqrt{121}$$

In other words:

$$10 < \sqrt{120} < 11$$

Instead of starting our guessing at 10.5, we start at 10.9 because 120 is really close to 121 (which is 11^2).

$10.9^2 = 118.81$ which is slightly too low. Now we know that our answer must be in between 10.9 and 11.

$$10.9 < \sqrt{120} < 11$$

Starting with 10.95, we see that $10.95^2 = 119.90$. Since this is too low, we change our guess to 10.96 and see that $10.96^2 = 120.12$.

10.95^2 is closer to 120 so we take that as our estimate.

$$\sqrt{120} \approx 10.95$$

The actual answer in this case is 10.954.

(d) Although this question may look slightly different (due to the decimal under the square root), the method we follow to estimate it is no different. First we find upper and lower bounds.

$$\sqrt{4} < \sqrt{4.5} < \sqrt{9}$$

This means that:

$$2 < \sqrt{4.5} < 3$$

For our first guess, we pick 2.5 and notice that $2.5^2 = 6.25$ which is too high. We continue going lower until we hit 2.1. $2.1^2 = 4.41$ which is still too low. But $2.2^2 = 4.84$ is too high. So our answer must lie somewhere in between.

$$2.1 < \sqrt{4.5} < 2.2$$

Since 2.1^2 is closer to 4.5 than 2.2^2 , our first guess needs to be close to 2.1. Let's start with 2.12. Checking the square gives $2.12^2 = 4.49$. Since this is too low, we check 2.13 whose square gives $2.13^2 = 4.53$.

Since 2.12^2 is closer to 4.5, we pick that as our estimate. So we write:

$$\sqrt{4.5} \approx 2.12$$

The actual answer in this case is 2.121.

- (e) As with one of the earlier questions we did, we need two estimates to solve this problem. First we need to estimate the fraction $\frac{3}{7}$.

Our upper and lower bounds are:

$$\frac{3}{9} < \frac{3}{7} < \frac{3}{6}$$

This means that:

$$0.33 < \frac{3}{7} < 0.5$$

Our answer needs to be in the above interval so we guess that it is 0.4.

Now we need to estimate $\sqrt{0.4}$. An interesting thing about numbers less than 1 is that their squares are **smaller** than them. For example, $0.25^2 = 0.0625$ etc. This means that our answer must be in between 0.4 and 1 - because only if you square a number less than 1 but greater than 0.4 will you get 0.4.

$$0.4 < \sqrt{0.4} < 1$$

We now take 0.5 as a guess. $0.5^2 = 0.25$ which is too low. Trying 0.6, we see that $0.6^2 = 0.36$. We also see that $0.7^2 = 0.49$. So our answer must be in between 0.6 and 0.7.

$$0.6 < \sqrt{0.4} < 0.7$$

Since 0.6^2 was closer to 0.4 than 0.7^2 , we start by making guesses close to 0.6 - say 0.61. We see that $0.61^2 = 0.37$. Going further up, we see that $0.62^2 = 0.38$ and $0.63^2 = 0.39$. Also, $0.64^2 = .41$. So we pick 0.63 as our final estimate.

$$\sqrt{\frac{3}{7}} \approx 0.63$$

The actual answer in this case is 0.654.

3. (a) 4.792×10^3
 (b) 1.87863×10^5
 (c) 1.2359×10^1
 (d) $1.90483203443217 \times 10^{14}$
4. To do this problem, we can first round up the base length from 12.89 *cm* to 13 *cm*. Using the Pythagorean theorem we get:

$$\begin{aligned} \sqrt{(13)^2 + (19)^2} &= \textit{Hypotenuse} \\ \sqrt{169 + 361} &= \sqrt{530} \textit{ cm} = \textit{Hypotenuse} \end{aligned}$$

All we need to do now is to estimate the square root. We first find upper and lower bounds:

$$\sqrt{529} < \sqrt{530} < \sqrt{576}$$

We got the upper and lower bounds by looking up perfect squares - you don't really need to have them memorized. So we get:

$$23 < \sqrt{530} < 24$$

529 is really close to 530 which means that $\sqrt{530}$ must be really close to 23. So for our first guess, we start with 23.1. Checking the square we see that $23.1^2 = 533.61$ which is too high.

So now we know that our answer must lie in between 23 and 23.1. In other words:

$$23 < \sqrt{530} < 23.1$$

We start with a guess right in the middle of our interval - 23.05. Checking the square we see that $23.05^2 = 531.3025$. Since this is too high, we go back by 0.01 and check 23.04. $23.04^2 = 530.8416$ which is still too high.

If we continue modifying our guess and checking the square in the above manner, we see that 23.02 is a good guess because $23.02^2 = 529.9204$ which is really close to 530.

So our estimate can be written as:

$$\sqrt{530} \approx 23.02$$

The actual length of the hypotenuse (if you used the Pythagorean theorem without rounding up any of the sides) is 22.95.

Our estimate was slightly higher than the actual answer because we rounded up one of the sides.

5. There are two ways to do this question - you could use the Pythagorean theorem and estimate the square root or you could notice an interesting fact about the triangle in the problem. Since we've estimated a lot of square roots so far, we're going to follow the second method.

Remember that the Pythagorean theorem says that:

$$(Base)^2 + (Height)^2 = Hypotenuse^2$$

Now, notice that we can round up both the base and height to 120 *units* and 90 *units* respectively. Look carefully at these numbers, they're related to each other. $120 = 4 \times 30$ whereas $90 = 3 \times 30$. In other words, both the base and the height are multiples of 30.

This means that our hypotenuse must also be a multiple of 30. Look at the Pythagorean theorem formula to understand this better.

A well known right triangle is the 3-4-5 triangle with base 3 *units*, height 4 *units* and a hypotenuse of 5 *units*.

Since our triangle has a base that's 3×30 *units* and a height that's 4×30 *units*, it means that our hypotenuse must be 5×30 *units*.

So we estimate that the length of the hypotenuse is:

$$Hypotenuse = 5 \times 30 = 150 \text{ units}$$

Note: Don't worry if you didn't completely get this way of figuring out the hypotenuse of a right triangle. You can get better at it by practicing and reading up on **Pythagorean triplets**.

6. In order for your scale to be balanced:

$$\textit{Weight of elephant} = \textit{Combined weight of 6th Graders}$$

So we need an estimate for the weight of an average elephant and another estimate for the weight of an average 6th Grader.

For the weight of an elephant, clearly 200 *kg* is too low (that's about 2 adult humans). At the same time, 10,000 *kg* is almost certainly too high. You might've seen videos of angry elephants overturning cars and buses. A typical car weighs about 1500 - 3000 *kg*, so if an elephant could push it over, the elephant should weigh about the same.

So let's say for the sake of argument that a typical elephant weighs 3000 *kg*.

Now there's no way that an average 6th Grader can weigh 10 *kg* - it's too little. There's also no way that the **average** weight of a 6th Grader is 80 *kg* (that's about the weight of an average full-sized adult).

Let's say that the weight of an average 6th Grader is somewhere in the middle - say 35 *kg*.

Note: It's important to realize that this is just **our estimate** for the average weight of a 6th Grader - there's no guarantee that it's actually this way. The point of doing this problem is to familiarize yourself with the thinking process, the numbers are not very important.

So now that we've got our two estimates, to find the number of 6th Graders, we divide:

$$\frac{\textit{Weight of an average elephant}}{\textit{Weight of an average 6th Grader}} = \frac{3000}{35} \approx 85$$

So we estimate that it'll take about 85 6th Graders to equal the weight of an average elephant.

This time, you're not given a right answer - the point of the problem was the method, not the answer. Solving the problem required you to break it up into two estimates. We're interested in your reasoning - the final answer doesn't need to be exactly right.

7. This is a relatively straightforward estimate. First we start by estimating the number of pizzas eaten by an average 6th Grader in a day. We make the assumption that all pizzas are the same size - i.e. a typical medium.

In a typical day, it's pretty safe to say that the average 6th Grader eats 0 pizzas.

Now we estimate the number of pizzas an average 6th Grader eats in a week. We'd still say that the number of pizzas eaten is about 0 or 1.

We keep going and try to estimate the number of pizzas eaten by an average 6th Grader in a month (i.e. 4 weeks). This would be about 4 times the number eaten in 1 week i.e. about 4 pizzas.

There are about $12 \text{ months} \times 4 \text{ weeks} = 50 \text{ weeks}$ in a year. This means that an average 6th Grader eats about $50 \times 4 = 200$ pizzas a year.

This estimate seems a bit too high. To correct it, let's say that an average 6th Grader eats an entire pizza every two weeks. This means that he/she eats about $50 \times 2 = 100$ pizzas a year.

Again, we are not given an actual answer to compare against. The point of doing these estimation problems is to practice mathematical reasoning. For example, in this problem you had to estimate the number of pizzas eaten in a day, a week and then extend that to a year. You also had to estimate the number of weeks in a year by assuming that each month had approximately 4 weeks.

8. This problem is very similar to our pizza problem. First, we need to estimate the number of chocolate bars in a typical Grade 6 classroom. To do this, we need to estimate the size of the class. Then, we need to think about arranging these chocolate bars and estimate the size of the resulting bar.

The first part is easy - a typical Grade 6 class is about 30 people. To estimate the number of chocolate bars in the class we could say that about half the class has a chocolate bar (except on Halloween where everyone does!). Remember, this estimate doesn't need to be exact.

So we've got about 15 chocolate bars. Now we have a problem - the question doesn't mention how to arrange the chocolate bars. In such situations, you're free to arrange them in whatever way you feel like. We're going to arrange them end to end and talk about how long the line of chocolate bars would be - try your own arrangement!

Now all we need is the length of a typical chocolate bar. 1 cm is obviously too small, whereas 30 cm is too big. So let's say that your average chocolate bar is about 15 cm - the same size as a standard ruler.

If we arranged our chocolate bars end to end, the length of our chocolate bar line would be:

$$\text{Length of chocolate bar line} = 15 \times 15 = 225 \text{ cm}$$

So our chocolate bar line would be about 2 meters long. That's about as tall as a pretty tall person.

Note that you could've done this question in many different ways, all leading to different answers. This is alright - focus on breaking the problem down instead of the final answer.

9. This question indirectly asks you to compare the sizes of an apple and the Earth.

First, we need to compare the size of an average human to that of the Earth. Then we'd be able to figure out how small we need to be such that the ratio of your new size to that of an apple is the same as your original size was to the Earth.

An average human is between 1 and 2 meters tall. In other words, the order of magnitude of the height of an average person is 0 ($1m = 10^0 m$).

The Earth has a diameter of about 13,000 *km* or $1.3 \times 10^7 m$ ($1 km = 10^3 m$).

So the ratio of the two sizes is:

$$\text{Human to Earth size ratio} = \frac{\text{Size of human}}{\text{Size of Earth}} = \frac{10^0}{10^7} = 10^{-7} m$$

Now the question is what is the size of the human, such that when the above ratio is taken with respect to an apple, it stays the same?

A typical apple is a few centimeters in diameter. In other words, it's about $10^{-2} m$ big - it has an order of magnitude of -2 .

Taking the ratio of sizes again:

$$\text{Shrunk Human to Apple size ratio} = \frac{\text{Size of shrunk human}}{\text{Size of apple}} = \frac{\text{Size of shrunk human}}{10^{-2}} = 10^{-7} m$$

We can rearrange this to get the size of the shrunk human:

$$\text{Size of shrunk human} = 10^{-7} \times \text{Size of apple} = 10^{-7} \times 10^{-2} = 10^{-7-2} = 10^{-9} m$$

$10^{-9} m$ is also known as a **nanometer**. It's about a thousand times smaller than most human cells. So you'd have to shrink yourself down by 9 orders of magnitude for the apple to be as big as the Earth to you.

10. To solve this problem, it might help to have look at a standard ballpoint pen. We're going to think of the pen as a cylinder and find it's volume - this should give us the amount of ink in it.

Then, we're going to assume that a straight line drawn with the pen is a super thin cuboid (a rectangular box, but ridiculously thin). Since the volume of this cuboid is known (it's just the volume of the cylinder), we can find it's length pretty easily.

To start, a typical ballpoint pen refill is about 10 *cm* (or 10^{-1} *m*) long and has a radius of about 1 *mm* (or 10^{-3} *m*). The volume of the refill is:

$$\text{Volume of refill} = \pi \times (\text{Radius})^2 \times (\text{Height}) = \pi \times (10^{-3})^2 \times 10^{-1} = 10^{-7}\pi \text{ m}^3$$

We don't really need to multiply by π - we can leave our answer in terms of it because we're doing an estimate.

The height (as measured from the paper) of a line drawn by a ballpoint pen is really thin. In fact, you can rarely feel it. If you've look at a standard box of papers, you'll see that the paper is about 0.01 *mm* (or 10^{-5} *m*) thick. The height of your line must be at least 10 times smaller because if you drew over your line 10 times, it still wouldn't be as tall as another sheet of paper.

This means that we can estimate the height of the line drawn by your pen to be around 10^{-8} *m*. What about the width or 'thickness' of the line? Well we know that if we drew about 10 lines very close to each other, we'd get a stack of lines about 1 *cm* wide. This means that we can estimate the width of each line to be about 10^{-3} *m* wide.

When we draw a line with our pen, we're actually depositing a small, thin layer of ink on the paper that's about 10^{-8} *m* tall and about 10^{-3} *m* wide. We know the length of this line depends on how big our refill is. So now we can find the maximum length of the line, as shown below:

$$\text{Volume of refill} = \text{Volume of cuboid} = (\text{Length}) \times (\text{Width}) \times (\text{Height})$$

We can rearrange this to get the length:

$$(\text{Length}) = \frac{\text{Volume of refill}}{(\text{Width}) \times (\text{Height})}$$

Plugging in the numbers we get:

$$(\text{Length}) = \frac{10^{-7}\pi}{10^{-3} \times 10^{-8}} = 10^4\pi \text{ m}$$

This is about 10 *km* or so. The well known ballpoint pen manufacturer Bic, claims that their pens can go upto 3 *km* before running out of ink. So our estimate is pretty close.