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Grade 7/8 Math Circles

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Estimation

Introduction

Reminder: Estimations do *not* have to be exact. There are many different ways that every estimation can be done. The solutions presented in this document are just one possible way of solving each problem.

If you ever find yourself without a calculator, a ruler, a computer, or if you are missing a few pieces of key information, solving math problems can become increasingly difficult. Often times, if you are in one of these situations, one of your options is to simply *guess* the answer.

Guessing can be effective at times, but an even more power tool which can be used in these situations is *estimation*!

Estimations can be done in just about any way that you want, as long as you can justify your reasoning. A few different ways you can estimate things, or different ways in which you can define an estimation are:

- A rough calculation.
- An educated guess.
- A combined sequence of assumptions.

The first thing that I want everyone to do today is *guess* how many of your handouts do think can fit into the box at the front of the room? Remember that a guess is different than an educated guess, or an estimation. Simply write down your initial thought without doing any kind of calculation.

Write your guess in the space below. Later into the class, we will try *estimating* how many handouts can fit into the box and see if we can get closer to the actual answer!

I guess that 1,000 (This is an example. It could be any number.) of these handouts can fit into the box at the front of the classroom.

This is an image of the box that we were discussing on page 1. You can use this image to help you solve a few of the upcoming problems in this lesson.

Estimation Techniques

The most common use for estimating is doing calculations without the help of a calculator, or even a pencil and paper.

Some of you might be comfortable with doing mental math already, but today we'll try doing some calculations that are almost *impossible* to do purely in your head. To solve questions like these, the best we can do is an estimation, to give us a rough idea of what the answer should actually be!

Today we'll try a few estimation techniques to do with:

- Finding the values of large exponents.
- Calculating areas.
- Estimating how much $1,000 of quarters would weigh.

Example 1:

Using purely mental math, quickly estimate the value of each of the calculations below.

(a) $248 \times 102 \approx 250 \times 100 = 25,000$ $\quad$ *Actual answer* $= 25,296$

(b) $1,584 \div 32 \approx 1,500 \div 30 = 50$ $\quad$ *Actual answer* $= 49.5$

(c) $\sqrt{957} \approx \sqrt{961} = 31$ $\quad$ *Actual answer* ≈ 30.94

Quickly check by using an estimation, which of these values *must* be *incorrect*.

(a) $3,575 \div 625 = 6.5 \longrightarrow \frac{3,575}{625} < \frac{3,600}{600} \longrightarrow \frac{3,575}{625} < 6 \longrightarrow$ *This* **must** *be incorrect*!

(b) $\sqrt{\pi} \approx 1.772 \longrightarrow \sqrt{2} < \sqrt{\pi} < \sqrt{4} \longrightarrow 1.414 < \sqrt{\pi} < 2 \longrightarrow$ *This might be correct*!

(c) $\frac{72}{15} \times \frac{21}{19} = 4.287 \longrightarrow \frac{72}{15} \times \frac{21}{19} > \frac{67.5}{15} \times \frac{19}{19} = 4.5 \longrightarrow$ *This* **must** *be incorrect*

Exponent Estimations

Before we get into an example of an estimation dealing with exponents, we first have to look in depth at exactly how exponents work. Consider doing the following calculation with any non-zero number 'a':

$$a^3 \times a^2 = (a \times a \times a) \times (a \times a) = a^5$$

Another way of looking at this calculation, is to simply *add* the values of the exponents together.

$$a^3 \times a^2 = a^{3+2} = a^5$$

Similarly to how we can *add* exponents together (if they have the same base number), we can also split an exponent into smaller pieces if we choose to do so.

$$a^7 = a^2 \times a^3 \times a^2$$

As a matter of fact, we can split up our exponents however we like! We can even split them into fractions, decimals or even negative numbers.

$$a^1 = a^{\frac{1}{2}} \times a^{\frac{1}{2}}$$

Judging from the equation above, $a^{\frac{1}{2}}$ must be the exact same thing as the square root of a!

There is one more exponent trick that we will look at today. Consider the following:

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$$

Another way of looking at this calculation, is to simply *multiply* the values of the exponents together.

$$(a^2)^3 = a^{2\times 3} = a^6$$

To summarize our results in general from the last page, we discovered these three properties:

- $a^b \times a^c = a^{b+c}$
- $a^{\frac{1}{2}} = \sqrt{a}$
- $(a^b)^c = a^{b \times c}$

Example 2:

Use the facts about exponents above to help you estimate the following:

(a) $\pi^6 \approx 3^6 = (3^2) \times (3^2) \times (3^2) = 9 \times 9 \times 9 = 81 \times 9 = 729$ $\qquad$ *Actual answer* ≈ 961.4

(b) $\pi^{20} \approx (\sqrt{10})^{20} = (10^{\frac{1}{2}})^{20} = 10^{\frac{1}{2} \times 20} = 10^{10} = 10\ \textit{billion}$ $\qquad$ *Actual answer* $\approx 8.77\ \textit{billion}$

(c) $2^{20} = (2^{10})^2 \approx (1,000)^2 = 1,000,000$ $\qquad$ *Actual answer* $= 1,048,576$

Scientific Notation

For questions like the ones in example 2, you end up getting numbers that are quite large. In fact, sometimes the numbers get so large, that even writing them down can take a *very* long time.

A 'googol' for example is an *extremely* large number. A googol starts with the number 1, and then has 100 zeros following it!

$$1\ Googol = 10000000000000000000000000000000000000000000000000000000...$$

The entire number can't even fit on the page! To fix this problem, we use something known as scientific notation.

What scientific notation does is take out as many powers of 10 from a number as possible, until you are left with a number between 1 and 10. In the case of a googol, there are 100 zeros, so that must mean there must be 100 powers of 10. Based on this, we can re-write a googol as:

$$1\ Googol = 1 \times 10^{100}$$

Here are a few examples of different numbers being written in scientific notation:

- $4,000,000 = 4 \times 10^6$
- $321 = 3.21 \times 10^2$
- $505,000,000,000 = 5.05 \times 10^{11}$

Example 3:

Try writing the following numbers in scientific notation:

(a) 7,250,000,000 $= 7.25 \times 10^9$

(b) 1,800 $= 1.8 \times 10^3$

Try the following calculations in scientific notation:

(a) $(5 \times 10^5) \times (4 \times 10^3) = 5 \times 4 \times 10^5 \times 10^3 = 20 \times 10^8 = 2 \times 10^9$

(b) $(8 \times 10^6) \div (2 \times 10^2) = \frac{8\times10^6}{2\times10^2} = \frac{8}{2} \times \frac{10^6}{10^2} = 4 \times 10^4$

Order of Magnitude Approximations

In estimations, we can use scientific notation to help us solve 'order of magnitude' or 'Fermi' approximation problems. The main point of these approximations is to get your answer to be within an *order of magnitude*.

An order of magnitude refers to the value of the exponent that a number has when written in scientific notation. For example, 2,000,000 is a number on the order of magnitude of 6.

$$2,000,000 = 2 \times 10^6$$

An example of a correct order of magnitude approximation for the size of this class would be 70 students. There are actually only 35 students enrolled in this class, but because both 35 *and* 70 are within an order of magnitude (i.e. neither of the numbers are 10 times bigger or smaller than the other), this would be considered an accurate order of magnitude approximation.

You can think of these calculations as being the answer to the question 'How many digits will be in my final answer?'. There will always be one more digit than the order of magnitude.

Order of Magnitude	Exponent	Value
0	10^0	1 – 9
1	10^1	10 – 99
2	10^2	100 – 999
3	10^3	1,000 – 9,999

One classic example of an order of magnitude approximation is, 'If every human on Earth held hands and formed a chain wrapping around the equator, how many times would they circle the Earth?'

Do you think this will be on the order of magnitude of 0? 1? 2? 3? More? Using a few facts, and a couple rough estimates, we should be able to find a reasonably accurate answer!

(i) First lets look at what information we know:

- There are about 7,000,000,000 or 7×10^9 people who live on Earth.
- The radius of the Earth is about 6,000 or 6×10^3 kilometers.
- The average human arm span is about 1 meter.

All three of the numbers above are not very accurate, but we can ignore the nitty-gritty details when considering order of magnitude approximations.

(ii) Now we just have to do some calculations!

$$\begin{aligned} Length\,Of\,Humans &= (7 \times 10^9)\,people \times \frac{1\,meter}{1\,person} \\ &= 7 \times 10^9\,meters \end{aligned}$$

$$\begin{aligned} Circumference\,Of\,Earth &= 2 \times \pi \times (6 \times 10^3)\,kilometers \\ &\approx 2 \times 3 \times (6 \times 10^3)\,kilometers \\ &\approx 4 \times 10^4\,kilometers \end{aligned}$$

(iii) If we take a ratio of these two distances, we can see how many times bigger one is than the other.

$$\frac{Length\,Of\,Humans}{Circumference\,Of\,Earth} = \frac{7 \times 10^9\,meters}{4 \times 10^4\,kilometers} = \frac{7 \times 10^6\,kilometers}{4 \times 10^4\,kilometers} \approx 2 \times 10^2 = 200$$

This means humans could approximately wrap around the Earth 200 times. The actual answer is closer to 350 when using more precise values.

Example 4:

This example has been done using the Math Circles classroom DC 1304

(a) Another example of an order of magnitude approximation would be 'How many Rubik's Cubes could fit inside this room?'

(i) Gather all of the information you can come up with.

- This is room is approximately the shape of a rectangular prism.
- This room approximately has a height of three meters, a width of eight meters and a length of five meters.
- About 16^3 or about 4,000 Rubik's Cubes could fit inside of a cubic meter.

(ii) From this information, I would estimate that about 400,000 Rubik's Cubes can fit inside of this room.

Volume of room $= Length \times Width \times Height \approx 3 \times 5 \times 8 = 120\, meters^3$

Number of cubes $\approx 120\, meters^3 \times \frac{4{,}000\, cubes}{meter^3} \approx 400,000\, cubes$

(b) One final example of an order of magnitude approximation would be 'What volume of water is lost by a leaky faucet every year?'

(i) Gather all of the information you can come up with.

A leaky faucet drips about 20 times every minute.

It would take about 2,000 drops of water to fill a 500 milliliter water bottle, so 1 liter is about 4,000 drops.

(ii) From this information, I would estimate that about 2,500 liters of water are wasted by a leaky faucet each year.

$$\frac{20\, drops}{1\, minute} \times \frac{60\, minutes}{1\, hour} \times \frac{24\, hours}{1\, day} \times \frac{365\, days}{1\, year} \approx 20 \times 60 \times 20 \times 400 \approx 10,000,000\, drops\ a\ year$$

Volume of wasted water $= 10,000,000\, \frac{drops}{year} \div 4,000\, \frac{drops}{liter} = 2,500\, liters$

Actual answer: Depending on how leaky the faucet is, you can waste anywhere between a 100, and up to 10,000 liters per year. The answer of 2,500 liters is quite average.

Now that we have tried a few of these types of problems, I want everyone to try *estimating* how many of your handout can fit in the box at the front of the room.

There are many different approaches! Here is one as an example:

- Each handout has 6 sheets of paper and 1 staple.
- It looks like 10 handouts stacked up would be about 1 cm. It also looks like the box is about 20 cm, therefore a stack of 200 handouts can fit inside the box!
- It looks like 2 stacks of paper could fit into the box.

Overall this tells us that 400 handouts should be able to fit into the box.

I estimate that about 400 of this handout could fit into the box at the front of the classroom.

Wisdom Of The Crowd

An interesting phenomenon is statistics is known as the *wisdom of the crowd* effect. What this property tells us, is that a large group of individuals' answers, when averaged together, are generally found to be more accurate than any given individual's single answer.

As an experiment, lets see if the average of all of our *guesses* is more accurate than our estimations when we take into account the wisdom of the crowd!

(*Use the space below to calculate the average of all of your classmates' guesses!*)

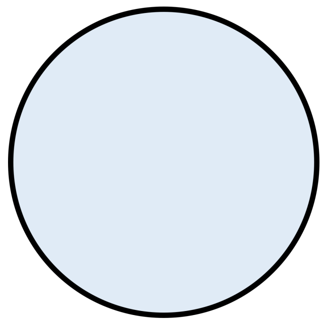
Number Of Handouts		
My Guess	My Estimation	Wisdom Of The Crowd

Estimating Areas

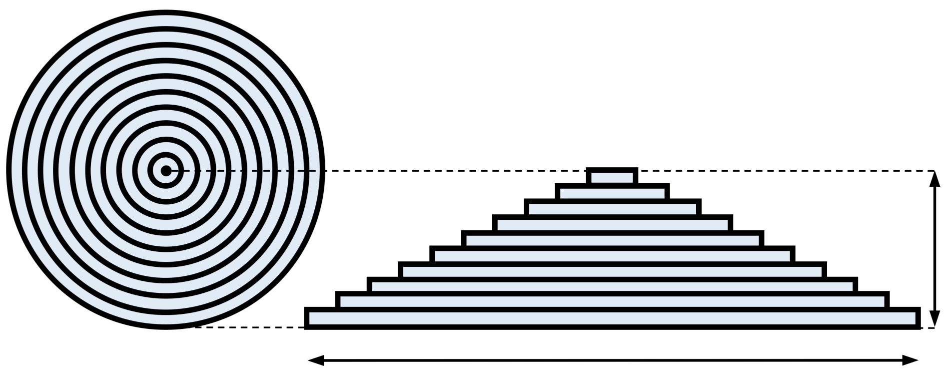
Now that we've covered estimating different kinds of arithmetic, we can finally try some more interesting things! In the next example, I will show you how you can derive the formula for the area of a circle ($A = \pi \times r^2$) using two different estimations!

Method 1:

(i) Consider the circle below. Imagine taking this circle and dividing it into many small concentric rings.



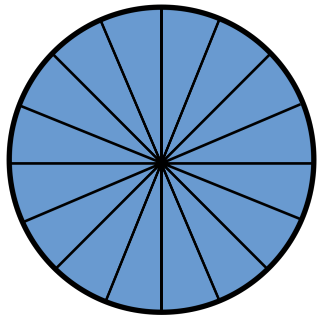
(ii) Picture taking each of these rings and unrolling them so they are perfectly straight. Each ring will approximately take the shape of a rectangle.



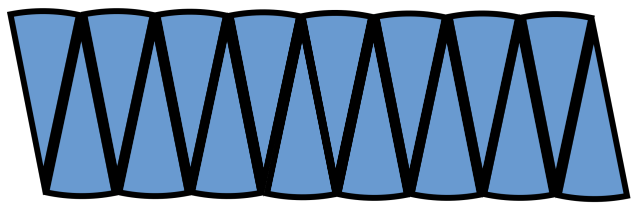
(iii) If you stack these rectangles on top of each other, they approximately form a triangle. The area of a triangle is $Area = \frac{1}{2} \times base \times height$. In this case, the height will be the radius of the circle, and the base will be the circumference of the circle. This gives us:

$$\begin{aligned} Area = \tfrac{1}{2} \times base \times height &= \tfrac{1}{2} \times R \times 2\pi R \\ &= \pi R^2 \end{aligned}$$

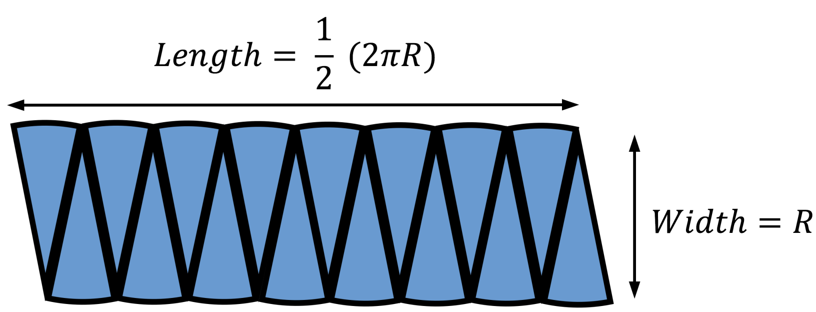
Method 2: (i) Imagine taking the same circle as in the previous method, except now instead of dividing it into rings, divide it into many small wedges.



(ii) Imagine taking these wedges, and arranging them to point vertically upwards and downwards in an alternating pattern as shown:



(iii) This approximately forms a rectangle. The width of the rectangle will be the radius of the circle. The length will be half the length of the circumference of the circle.



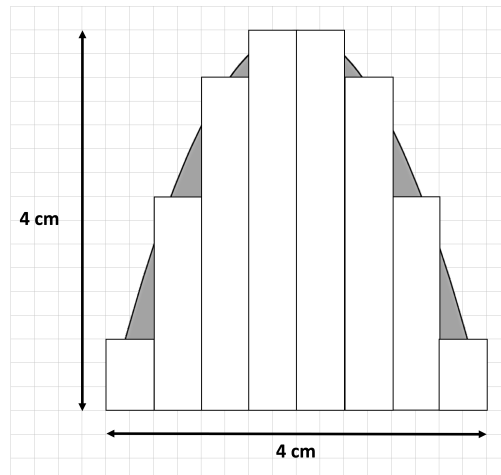
(iv) Calculating the area of this 'rectangle' gives us:

$$\begin{aligned} Area &= Length \times Width \\ &= \tfrac{1}{2} \times 2\pi R \times R \\ &= \pi R^2 \end{aligned}$$

Example 5:

Using similar ideas to the ones used in the area of a circle approximations, estimate the area of the following shape:

We can estimate the area of this shape by filling it with many small rectangles as seen to the right.



If we add up the area of each of these rectangles, it should approximately give us the area of the original shape.

The thinner you make your rectangles, and the more rectangles that you use can make your estimate more precise.

$$Area \approx Area\ Of\ Each\ Rectangle$$

$Area \approx 2(0.5 \times 0.75) + 2(0.5 \times 2.25) + 2(0.5 \times 3.5) + 2(0.5 \times 4)$
$Area \approx 0.75 + 2.25 + 3.5 + 4$
$Area \approx 10.5\ cm^2 \longrightarrow Actual\ answer = 10\frac{2}{3}\ cm^2$

Problems

1. How many digits would each of these numbers have?

(a) $(\sqrt{10})^{10} = (10^{\frac{1}{2}})^{10} = 10^{\frac{1}{2}\times 10} = 10^5 = 100,000 \longrightarrow 6\ digits$

(b) $5^{10} = (\frac{10}{2})^{10} = \frac{10^{10}}{2^{10}} \approx \frac{10^{10}}{10^3} \approx 10^7 \approx 10,000,000 \longrightarrow$ This was an *overestimate.* The actual answer should be slightly *less*, so it will actually only have 7 digits.

(c) $2^{500} = (2^{10})^{50} \approx (10^3)^{50} \approx 10^{3\times 50} \approx 10^{150} \longrightarrow 151\ digits$

(d) $\pi^{124} \approx (10^{\frac{1}{2}})^{124} \approx 10^{\frac{1}{2}\times 124} \approx 10^{62} \longrightarrow$ From this estimation, we get an answer of 63 digits. The actual answer is 62 digits, which is within an order of magnitude, so it is a reasonable estimation.

(e) $7^{12} = (7^2)^6 \approx 50^6 \approx (\frac{100}{2})^6 \approx (\frac{10^2}{2})^6 \approx \frac{10^{12}}{2^6} \approx \frac{10^{12}}{10^2} \approx 10^{10} \longrightarrow 11\ digits$

2. What will be the order of magnitude for each of these calculations? You can use any facts, guesses and estimates that seem reasonable to you.

Your order of magnitude will change based on what units your final answer is in!

(a) If you drew one long straight line with a new pen, how long would it be when you run out of ink?

Every 2 months at school, I usually need to start using a new pen because my old one has ran out of ink. Every school day I usually write down about 5 pages of notes (double sided) with my pen. Each page probably has about 500 words, and each word probably uses about 5 centimeters of ink.

Combining all of these assumptions gives us:

$$\frac{5\ cm}{1\ word} \times \frac{500\ words}{1\ page} \times \frac{5\ pages}{1\ school\ day} \times \frac{40\ school\ days}{2\ months} \times \frac{2\ months}{1\ pen} = \frac{500{,}000\ cm}{1\ pen} = 5\ kilometers\ per\ pen$$

An average pen actually has a writing distance of 2 kilometers, so this estimate is on the same order of magnitude. (Order of magnitude of 0)

(b) How many times would you have to fold a sheet of paper in half for it to reach to the moon?

An average package of 500 sheets of paper is about 5 centimeters. From this, if we multiply our amount of paper by 20, we find that 10,000 sheets should be about 1 meter.

The distance to the moon is a value you can find in a book or online. It is about 400,000 kilometers, or equivalently, 400,000,000 meters.

$$400,000,000\ meters \times \frac{10{,}000\ sheets}{1\ meter} = 4,000,000,000,000\ sheets\ of\ paper$$

Now by using some exponent rules, we can find that:

$$4,000,000,000,000 = 4 \times 1,000 \times 1,000 \times 1,000 \times 1,000 \approx 2^2 \times 2^{10} \times 2^{10} \times 2^{10} \times 2^{10} = 2^{42}$$

So this tells us we need about 2^{42} sheets of paper. We also know that every time a sheet of paper is folded in half, its height should double. This allows us to say that a sheet of paper folded in half 42 times, should be just as high as a stack of 2^{42} sheets of paper!

The actual answer is the same as the estimation, about 42 times. (Order of magnitude of 1)

(c) How fast does human hair grow in centimeters per day?

I get a haircut once every 2 months, or about every 60 days. Every haircut I get trims off about 4 cm of hair.

$$\frac{4\,centimeters}{60\,days} = 0.0\overline{6}\,centimeters\,per\,day$$

The actual answer is about 0.04 centimeters per day. (Order of magnitude of -2)

(d) What is the mass of the Earth?

The Earth is made mostly out of heavy rocks. I know water has a density of 1,000 kilograms per cubic meter, so I will assume the rocks inside the Earth are about 4 times more dense (4,000 kilograms per cubic meter).

The Earth is a sphere with a radius of about 6,000 kilometers, or 6,000,000 meters, so I can calculate its volume :

$$Volume = \frac{4}{3}\pi R^3 \approx \frac{4}{3} \times 3 \times 6,000,000^3 \approx 10^{21}\,cubic\,meters$$

$$Mass = 10^{21}\,cubic\,meters \times 4,000\,kilograms\,per\,cubic\,meter = 4 \times 10^{24}\,kilograms$$

The actual answer is about 6×10^{24} *kilograms*. (Order of magnitude of 24)

(e) How many trees are there on the entire planet?

The radius of the Earth is about 6,000 kilometers, so the surface area of the Earth can be found;

$$SurfaceArea = 4\pi R^2 \approx 4{\times}3{\times}6,000^2 \approx 12{\times}6{\times}6{\times}10^6 \approx 70{\times}6{\times}10^6 \approx 420,000,000km^2$$

Also, I know only about one quarter of the surface area of Earth is land.

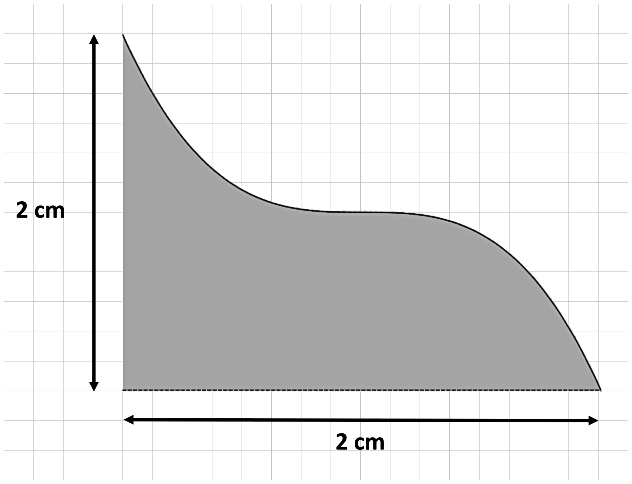
$$Land\ Area = \frac{432,000,000\,km^2}{4} \approx 100,000,000km^2$$

Assuming there are about 10,000 trees per square kilometer on average, we get:

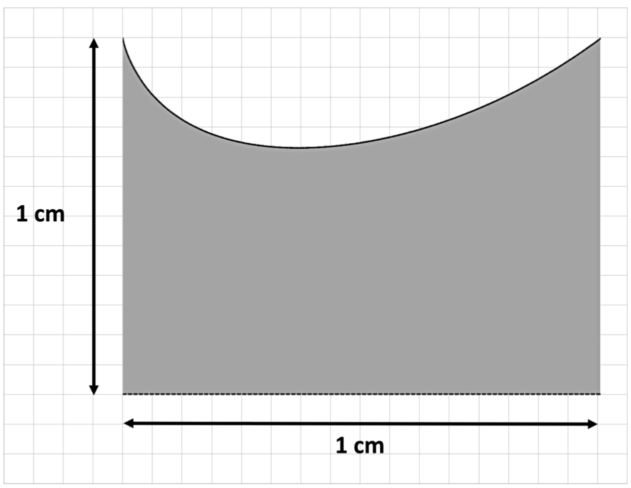
$$Total\ Trees \approx 100,000,000\,km^2 \times 10,000\,trees\,per\,km^2 = 10^{12}\,trees$$

there are actually about 3 trillion, or 3×10^{12}. (Order of magnitude of 12)

3. Estimate the areas of the following shapes:

(a) 

The actual area of this shape is exactly 2 cm^2.

(b) 

The actual area of this shape is about 0.83 cm^2.