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Grade 7/8 Math Circles

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The Scale of Numbers

Introduction

Last week we quickly took a look at scientific notation, which is one way we can write down really *big* numbers. We can also use scientific notation to write very *small* numbers.

$$\begin{aligned}
1 \times 10^{3} &= 1,000 \\
1 \times 10^{2} &= 100 \\
1 \times 10^{1} &= 10 \\
1 \times 10^{0} &= 1 \\
1 \times 10^{-1} &= 0.1 \\
1 \times 10^{-2} &= 0.01 \\
1 \times 10^{-3} &= 0.001
\end{aligned}$$

As you can see above, every time the value of the exponent decreases, the number gets smaller by a factor of 10. This pattern continues even into negative exponent values!

Another way of picturing negative exponents is as a *division* by a *positive* exponent.

$$10^{-6} = \frac{1}{10^{6}} = 0.000001$$

In this lesson we will be looking at some famous, interesting, or important small numbers, and begin slowly working our way up to the biggest numbers ever used in mathematics!

Obviously we can come up with any arbitrary number that is either *extremely small* or *extremely large*, but the purpose of this lesson is to only look at numbers with some kind of mathematical or scientific significance.

Extremely Small Numbers

1. **Zero**

   - Zero or '0' is the number that represents nothingness. It is the number with the smallest *magnitude*.
   - Zero only began being used as a number around the year 500. Before this, ancient mathematicians struggled with the concept of '*nothing*' being '*something*'.

2. **Planck's Constant**

   This is the smallest number that we will be looking at today other than zero.

   - Planck's constant is labelled as 'h' and has a value of $h = 6.626 \times 10^{-34} \, \frac{kg\,m^2}{s}$.
   - Planck's constant is the first of three universal constants that we will discuss today.
   - Planck's constant can be defined as the ratio between a light ray's energy and its frequency.

   $$h = \frac{Energy}{Frequency}$$

3. **The Mass of an Electron**

   - The mass of an electron 'm_e' is about $9.11 \times 10^{-31}\ kg$.
   - Electrons are thought to be '*fundamental particles*', meaning that they cannot be broken down into smaller pieces.

4. **The Mass of a Proton**

   - The mass of a proton 'm_p' is about $1.67 \times 10^{-27}\ kg$.
   - Protons, along with neutrons are the tiny particles which make up the nuclei of atoms!
   - A single proton has about as much mass as 1,830 electrons!

Very Small Numbers

1. **The Gravitational Constant**
   - The gravitational constant is labelled 'G' and has a value of $G = 6.67 \times 10^{-11}\ \frac{m^3}{kg\ s^2}$.
   - This value is what determines how strong gravity is throughout the entire universe!
2. **Your Chances of Getting a Royal Flush**

Example 1:

When drawing 5 consecutive cards (without replacement) from a deck of cards, what is the probability of drawing a royal flush (10, Jack, Queen, King and Ace of the same suit)?

Small Numbers

1. **The Square Root of Two**
   - The square root of 2, or '$\sqrt{2}$' is one of the most famous *irrational* numbers.
   - The square root of 2 has a value of approximately $\sqrt{2} = 1.4142136$.
   - The diagonal of a square can always be found by multiplying its side length by $\sqrt{2}$!

How can we prove that $\sqrt{2}$ is irrational? First ask, 'what exactly is a *rational* number?'

A rational number is *any* number that can be written as a fraction $\frac{p}{q}$ where p is an integer and q is a non zero integer.

$$\frac{3}{16} = 0.1875 \qquad \frac{1}{7} = 0.\overline{142857} \qquad \frac{9}{4} = 2.25$$

Let's see what happens if we try to write $\sqrt{2}$ as a fraction.

If we can write $\sqrt{2}$ as a fraction, we can say it must have a numerator a and a non zero denominator b, where a and b have no positive *common factor* other than 1 (so the fraction is irreducible).

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides of this equation, and doing some rearrangements gives us:

$$\begin{aligned}\sqrt{2} &= \frac{a}{b}\\ 2 &= \frac{a^2}{b^2}\\ 2b^2 &= a^2\end{aligned}$$

Because a^2 is two times a number, that must mean that a^2, and a itself are even.

Now that we know a is even, we can write it as any general even number $a = 2n$.

$$\begin{aligned}2b^2 &= a^2\\ 2b^2 &= (2n)^2\\ 2b^2 &= 4n^2\\ b^2 &= 2n^2\end{aligned}$$

Now from this relation we can say that b is *also* an even number!

Because a and b are both *even* numbers, a and b actually have a common factor greater than 1. This means that we *contradicted* our original statement, meaning $\sqrt{2}$ can *not* be written as a *non reducible* fraction, i.e. it must be irrational.

2. **The Golden Ratio**

- The Golden Ratio ϕ has a value of $\frac{\sqrt{5}+1}{2} \approx 1.618033989$.
- The Golden Ratio appears all over the place in the real world (shells, trees, animals etc).
- One place where you can find the Golden Ratio in mathematics is in the '*Fibonacci Sequence*'.

The Fibonacci Sequence is a sequence where the first 2 terms are *both* 1. Each consecutive term after the first two will always be the *sum* of the two *previous* terms.

If you take a ratio of 2 adjacent terms in the fibonacci sequence (the bigger one divided by the smaller one), you will get an *estimation* for the golden ratio. The further you go into the sequence, the more accurate this estimation will become.

The Fibonacci Sequence $\longrightarrow 1, 1, 2, 3, 5, 8.....$

Example 2:

(a) Write out the first 12 terms of the Fibonacci Sequence.

(b) Estimate the Golden Ratio by taking a ratio of the following Fibonacci Sequence terms:

(i) The 6^{th} and 5^{th} terms.

(ii) The 12^{th} and 11^{th} terms.

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3. **Two**

   - A well known fact is that two or '2' is actually the *first* prime number (rather than 1). The reasoning behind this has to do with something known as '*The Fundamental Theorem of Arithmetic*'.

The Fundamental Theorem of Arithmetic $\longrightarrow$ "*Every integer greater than one is either prime itself, or it can be written as a **unique** product of primes.*"

4. **Four**

   - Four or '4' is the first *composite* number, meaning it has factors other than 1 and itself.
   - Four is the only number in the English alphabet that is equal to the number of letters in its name!

5. **Six**

   - Six or '6' is the first *perfect* number. This means that the sum of its proper positive divisors is equal to the number itself.

$$6 = 1 + 2 + 3$$

Large Numbers

1. **Three Hundred and Sixty**

   - Three hundred and sixty or '360' is the number of degrees within a circle.
   - 360 is a *highly composite number*, meaning it is a positive integer with more divisors than any positive integer less than it!

   Divisors of 360 $\longrightarrow$ 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360

2. **One Thousand Eight Hundred and Eighty One**

   - One thousand eight hundred and eighty one or '1,881' is a *palindromic* number, meaning it is the same when written either forwards or backwards.
   - 1,881 is also a *strobogrammatic* number, meaning it is the same when written upside down or right side up.

Very Large Numbers

1. **The Speed of Light**

   - The speed of light 'c' is currently understood to be the fastest possible speed that *anything* can travel at.
   - The speed of light is about $c = 3 \times 10^8$ meters per second. This is fast enough to go around the entire planet 7.5 times per second!

2. **The Distance to the Sun**

   - The Sun is a distance of 1.5×10^{11} meters away. This distance is called the '*astronomical unit*' or the AU for short.
   - If you were to drive a car to the sun, it would take approximately 170 years to arrive.

3. **The Number of Ways to Arrange the Alphabet**

Example 3:

How many ways are there to arrange the 26 letters in the English alphabet? Write your answer in scientific notation.

Ridiculously Large Numbers

1. **The Largest Known Prime Number**

   - The current record for the largest known prime number is $2^{74,207,281} - 1$
   - This prime number has a staggering 22,338,618 digits! At 3,000 digits per page, it would take a whopping 7,446 pages to write down this number.
   - A '*Mersenne Prime*' is a prime number 'M_n' such that $M_n = 2^n - 1$ where n is a positive integer.

2. **A Googolplex**

   - The classic example of a massive number is a googol, which is equal to 10^{100} or a 1 followed by 100 zeros. The classic example of a ridiculously large number is a googolplex, which is a 1 followed by a googol of zeros ($10^{\,a\ googol}$), or $10^{10^{100}}$.

   Can we even imagine how big of a number this is? Lets try to!

   A googolplex has 10^{100} zeros. If we say each of these zeros can be written in a space of 1 centimeter, how long would the entire number have to be? It would have to be ______________ long!

Example 4:

How long would the digits of a googolplex be in terms of the following units:

(a) Meters?

(b) Kilometers?

(c) Lightyears (The distance light can travel in an entire year)?

Unfathomable, Mind Numbingly Large Numbers

Before we talk about these numbers, I have to warn you not to think too hard about them. If you actually processed every digit of these numbers in your brain, your head would turn into a black hole. *Seriously.*

To describe these unimaginably large numbers, we actually need to first learn a new kind of notation. Exponents, and even 'exponent towers' like seen in a googolplex, can not grasp the enormity of these numbers. What we have to use is something known as '*Knuth's up arrow notation*'.

Knuth's Up Arrow Notation

To understand this new notation we will actually start by describing what multiplication is.

$$3 \times 4 = 3 + 3 + 3 + 3$$

As you can see above, 3 multiplied by 4 is just another way of saying "add four threes together".

To represent bigger numbers we use exponentiation.

$$3^4 = 3 \times 3 \times 3 \times 3$$

As you can see above, 3 to the power of 4 is just another way of saying "multiply four threes together".

To represent even bigger numbers, we have to use something called '*tetration*'. Tetration is where we will introduce Knuth's up arrow notation.

$$3 \uparrow\uparrow 4 = 3^{3^{3^3}}$$

As you can see, '3 arrow arrow 4' is just another way of saying "exponentiate four threes together".

Multiplication $\longrightarrow$ Repeated addition.

Exponentiation $\longrightarrow$ Repeated multiplication.

Tetration $\longrightarrow$ Repeated exponentiation.

Example 5:

For the expression below, what value of n is needed for the resulting number to be larger than a googolplex?

$$3 \uparrow\uparrow n$$

Now we have only scratched the surface of the capabilities of up arrow notation. We can actually take this process even further! Consider the following:

$$3 \uparrow\uparrow\uparrow 4$$

We know that in this new notation, 2 arrows represents tetration, or repeated exponentiation. 3 arrows represents '*pentation*' which is repeated tetration.

$$3 \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3 \uparrow\uparrow 3$$

'3 arrow arrow arrow 4' is just another way of saying "tetrate four threes together". This number is so absurd that there's no way to even begin to think how big it is.

Graham's Number

This number, known as 'Graham's Number', was the Guinness World Record holder for the largest number ever used to actually solve a problem in 1980. Since then it has been passed by other monstrous numbers like TREE(3), but we're not going to talk about those today.

To begin to see the size of Graham's number we will start with the following:

$$3 \uparrow\uparrow 3 = 3^{3^3} = 3^{27} = 7,625,597,484,987$$

Already we have a very large number, but we are not even *remotely* close to Graham's Number yet. Next consider:

$$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow 3 \uparrow\uparrow 3$$

This step is *pentation* as we saw on the last page. Let's try to see what this will equal.

$$3 \uparrow\uparrow 3 \uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow 7,625,597,484,987$$

What is this saying? We know two arrows means tetration, which is the same as repeated exponentiation, so that must mean we have to exponentiate threes together 7,625,597,484,987 times!!!

$$3^{3^{3^{3^{3^{3^{3^{3^{\cdot^{\cdot^{\cdot^{3}}}}}}}}}}} \quad \longrightarrow \quad (A\,tower\,of\;7,625,597,484,987\;threes!)$$

This is by far the biggest number so far in this lesson, but it's still not even close to Graham's number. The next number we will look at is called '$g1$'.

$$g1 = 3 \uparrow\uparrow\uparrow\uparrow 3$$

Now we have *four* arrows, which would represent *hexation*, or repeated pentation. Don't bother trying to imagine this number at all, just know that it is extremely large, even when compared to $3 \uparrow\uparrow\uparrow 3$.

The next number '$g2$' is where things start to get very interesting. This number will have an amount of $g1$ *arrows*!

$$g2 = 3 \uparrow\uparrow\uparrow\uparrow\uparrow \ldots \uparrow\uparrow\uparrow 3 \quad \longrightarrow \quad (Total\;of\;g1\;arrows)$$

If your mind hasn't been boggled yet, the next number we will look at is '$g3$'. Similarly to $g2$, $g3$ will be a number with $g2$ arrows.

$$g3 = 3\uparrow\uparrow\uparrow\uparrow\uparrow\ldots\uparrow\uparrow\uparrow 3 \quad \longrightarrow \quad (Total\ of\ g2\ arrows)$$

Finally, if we keep repeating this pattern, Graham's Number G is equal to $g64$.

$$\boldsymbol{G} \;=\; \left.\begin{matrix} \underbrace{\mathbf{3}\uparrow\uparrow\cdots\cdots\cdots\cdots\cdots\uparrow\mathbf{3}} \\ \underbrace{\mathbf{3}\uparrow\uparrow\cdots\cdots\cdots\cdots\uparrow\mathbf{3}} \\ \vdots \\ \underbrace{\qquad\qquad\qquad} \\ \underbrace{\mathbf{3}\uparrow\uparrow\cdots\cdots\uparrow\mathbf{3}} \\ \mathbf{3}\uparrow\uparrow\uparrow\uparrow\mathbf{3} \end{matrix}\right\} \textbf{64 layers}$$

This number is so big that it is impossible to even begin to comprehend it. One important thing to remember though, is that Graham's number is still equal to exactly *zero* percent of infinity.

Problems

1. Find the 2^{nd} perfect number. Show that it is perfect.

2. How many Mersenne Primes are there that have a value less than 1,000. What numbers are they?

3. What are the first 5 highly composite numbers? How many divisors does each have?

4. Is $\sqrt{8}$ irrational? How do you know? Show your work.

5. Using $\sqrt{2}$, 0.5, 10, π and any *BEDMAS* operations at most once each:

   (a) What is the largest number you can make?

   (b) What is the number with the smallest *magnitude* that you can make?

6. 1,881 was the most recent *palindromic* and *strobogrammatic* year. What will be the next *palindromic* and *strobogrammatic* year?

7. Evaluate each of the following:

   (a) $2 \uparrow\uparrow 2$

   (b) $2 \uparrow\uparrow\uparrow 2$

   (c) $2 \uparrow\uparrow\uparrow\uparrow 2$

   (d) $9 \uparrow\uparrow 3$

8. To see how fast numbers can grow when using Knuth's up arrow notation, fill out the following table:

   The 'Growth Factor' is how many times bigger each term is than the previous term.

Arrow Notation	Standard or Scientific Notation	Growth Factor
$2 \uparrow\uparrow 0$	1	N/A
$2 \uparrow\uparrow 1$	2	2
$2 \uparrow\uparrow 2$	4	2
$2 \uparrow\uparrow 3$		
$2 \uparrow\uparrow 4$		
$2 \uparrow\uparrow 5$		

(You may need the help of a computer for the last row!)

9. Using some combination of G the gravitational constant, c the speed of light, and $\hbar$ the *reduced* Planck's constant, find a universal length in terms of meters through the use of unit analysis.

$$\hbar = \frac{h}{2\pi} = 1.055{\times}10^{-34}\frac{m^2\,kg}{s}$$

$$G = 6.67{\times}10^{-11}\frac{m^3}{kg\,s^2}$$

$$c = 3{\times}10^{8}\frac{m}{s}$$

10. How many more *zeros* does a googolplex have than $10^{10^{10}}$?