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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

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Math Jeopardy

Series and Polygonal Numbers

$100 What is the sum of the natural numbers from 21 to 519?

For *natural* numbers, our sum can be written as:

$$Sum = \frac{(First\,Term + Last\,Term)(Number\,Of\,Terms)}{2} = \frac{540 \times 499}{2} = 134,730$$

$200 What is the sum of the odd numbers from 1 to 1001?

For the first n *odd* numbers, our sum can be written as:

$$Sum = n^2$$

So how many odd numbers do we have? Remember that any odd number can be written as $2n - 1$ where n is the number that tells you which odd you have. For example, if $n = 4$, you will have the fourth odd number.

Clearly 1 is the first odd number, but what odd number is 1,001? Let's check by using the $2n - 1$ equation.

$$\begin{aligned} 1,001 &= 2n - 1 \\ 1,002 &= 2n \\ 501 &= n \end{aligned}$$

So 1,001 is the 501^{st} odd number. That means from the 1^{st} odd number, to the 501^{st} odd number, we will have a total of 501 odd numbers!

The sum will then be:

$$Sum = n^2 = 501^2 = 251,001$$

$300 True or False? The triangular numbers follow the pattern: odd odd, even, even odd, odd, even, even...etc.

True!

Remember that triangular numbers come from the sum of the first n natural numbers. This means the 4^{th} triangular number will be the sum of the first 4 natural numbers ($T_4 = 1 + 2 + 3 + 4 = 10$).

In general we can write this as:

$$T_n = 1 + 2 + 3 + 4 + ... + n$$

Next we can say that the natural numbers alternate between even and odd numbers. We also know that when you take a number, and add an *even* number to it, its parity will not change. This means if its even, it will still be even. If it's odd, it will still be odd.

Alternatively, if you take a number, and add an *odd* number to it, its parity *will* change.

Finally, because every odd number is *two* spaces away from the previous odd number, we can say that the parity of every triangular number should change after every *two* triangular numbers (A.K.A. odd, odd, even, even, odd, odd...etc.)

$400 Find two square numbers that are also triangular numbers (*Excluding zero!*).

To solve this question, the easiest way is to write out the first few square numbers and the first few triangular numbers, and then check if any of the square numbers have the same value as any of the triangular numbers.

Remember that any square numbers can be written as n^2 and any triangular number can be written as $\frac{n(n+1)}{2}$.

n	n^2	$\dfrac{n(n+1)}{2}$
1	1	1
2	4	3
3	9	6
4	16	10
6	36	15
7	49	21
8	64	28
9	81	36

From this table, you can see that the 1^{st} triangular and the 1^{st} square number both have the same value of 1. The 6^{th} square number and 9^{th} triangular number have the same value of 36. Therefore 1 and 36 are both square and triangular numbers.

$500 What is the general sum of any two consecutive triangular numbers T_n and T_{n+1}? Make sure you simplify your answer!

We already know that a triangular number T_n can be written as:

$$T_n = \frac{n(n+1)}{2}$$

If we want to write the next triangular number *after* T_n (which we can call T_{n+1}), we simply need to replace every n with an $n+1$.

$$T_{n+1} = \frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$$

The sum of these two triangular numbers will then be:

$$T_n + T_{n+1} = \frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}$$

$$= \frac{n(n+1) + (n+1)(n+2)}{2}$$

$$= \frac{n^2 + n + n^2 + 2n + n + 2}{2}$$

$$= \frac{2n^2 + 4n + 2}{2}$$

$$= n^2 + 2n + 1$$

$$= (n+1)^2$$

Therefore, the sum of two consecutive triangular numbers can be written as:

$$T_n + T_{n+1} = (n+1)^2$$

Angles and Circles

$100 What is the circumference of a semi-circle if its diameter is 4 units?

The equation for the circumference of a full circle is:

$$C_{Full\,Circle} = \pi \times diameter$$

For a semicircle, we will have half of the circumference of a full circle:

$$C_{Semi\,Circle} = \frac{1}{2}\pi \times diameter$$

For a diameter of 4 units, this gives us:

$$C_{Semi\,Circle} = \frac{1}{2}\pi \times diameter = \frac{1}{2}\pi \times 4 = 2\pi$$

So this semi circle will have a circumference of 2π units.

$200 If a circle had 400° instead of 360°, what would be the equivalent to 45°?

First check what fraction 45° is of 360°:

$$\frac{45^\circ}{360^\circ} = \frac{1}{8}$$

Now if we defined a circle to have 400° instead of 360°, the equivalent to 45° would have to be one eighth of 400°.

$$400^\circ \times \frac{1}{8} = 50^\circ$$

Therefore the equivalent to 45° in this newly defined circle would be 50°.

$300 Convert $\frac{270}{\pi}^\circ$ to radians.

To convert between radians and degrees we can use this equation:

$$Radians = Degrees \times \frac{\pi}{180^\circ}$$

For an angle of $\frac{270}{\pi}^\circ$, this will give us:

$$Radians = Degrees \times \frac{\pi}{180^\circ} = \frac{270^\circ}{\pi} \times \frac{\pi}{180^\circ} = 1.5$$

So $\frac{270}{\pi}^\circ$ is equivalent to exactly 1.5 radians.

$400 What is the length of an arc that subtends an angle of 3 radians from the center of a circle with a radius of 2.5 units?

The equation used to calculate arc length is:

$$Arc\ Length = r\theta$$

Where r is the radius of the circle and θ is the angle in terms of radians.

For a radius of 2.5 units and an angle of 3 radians, we get:

$$Arc\ Length = r\theta = 2.5 \times 3 = 7.5$$

Therefore this arc will have an arc length of 7.5 units.

$500 How many full rotations does a wheel of radius 22" make if it comes to a stop 150 meters from its starting point?

The first thing to do in this question is to convert all of our units into the *same* units.

$$Radius = 22\text{"} \times \frac{2.54\,cm}{1\text{"}} = 55.88\,cm = 0.5588\,meters$$

So the radius of the wheel can be written as 0.5588 meters.

Next we know that for each rotation the wheel makes, it will travel forwards a distance equal to the wheel's circumference.

$$Circumference = 2\pi R = 2\pi \times 0.5588 \approx 3.511\,meters$$

Now we just have to see how many of these circumferences it takes to make 150 meters.

$$\frac{150\,meters}{3.511\,meters\ per\ circumference} \approx 42.7\,circumferences$$

So the wheel will make 42 full rotations.

Probability

\$100 When rolling a 27-sided die (with the numbers 1 to 27), what is the probability of getting a number less than 9?

Overall there are 27 numbers, and we are equally likely to roll any of these 27 numbers. Also we know that there are 8 numbers with a value less than 9 (1, 2, 3, 4, 5, 6, 7 and 8). This will give us a probability of:

$$Probability = \frac{\#\,Of\ Favorable\ Outcomes}{\#\,Of\ Total\ Outcomes} = \frac{8}{27}\ or\ 29.6\%$$

\$200 What is the probability of flipping a coin 5 times, and getting 3 heads, a tail and then another head, in that order?

Getting this sequence of heads and tails (HHHTH) is one unique possible outcome when flipping a coin five times (i.e. there is only one way to get to this outcome).

Overall, we know that there will be 2^5 or 32 total outcomes, because each flip of the coin gives 2 possible outcomes, and there are 5 flips overall. This means the probability will be:

$$Probability = \frac{\#\,Of\ Favorable\ Outcomes}{\#\,Of\ Total\ Outcomes} = \frac{1}{32}\ or\ 3.125\%$$

\$300 If you roll a 6-sided die twice, what is the probability that you will get two numbers that add up to 8?

When rolling a 6-sided die twice, there will be a total of 6^2 or 36 possible outcomes, because each of the two rolls will have 6 possibilities.

Using the numbers 1, 2, 3, 4, 5 and 6, there are a total of 5 ways that we can add two of these numbers to give a result of 8.

$$2 + 6 = 8$$
$$3 + 5 = 8$$
$$4 + 4 = 8$$
$$5 + 3 = 8$$
$$6 + 2 = 8$$

This gives us a probability of:

$$Probability = \frac{\#\,Of\ Favorable\ Outcomes}{\#\,Of\ Total\ Outcomes} = \frac{5}{36}\ or\ 13.\overline{8}\%$$

\$400 A medical test has a 40% chance of detecting a disease correctly. If 100 people are tested at random, how many have the disease?

This question does not give enough information for you to be able to correctly solve it, so the correct answer is "I don't know."

This would be similar to asking, "There is an 80% chance of rain tomorrow. Will it rain tomorrow?". You can't say yes or no for sure. It is likely to rain, but it could also not rain at all, so there is no definite answer to this question.

\$500 In a room of 25 people, what probability is the closest to the probability of two people sharing a birthday?

To start, lets look at a simpler case of having a room with just 3 people.

The first person can have his/her birthday on any of the 365 days of the year, so there is a 100% chance that he/she will not share a birthday with anyone else.

The second person can have his/her birthday on any of the 365 days of the year, *except* for the day that the first person has their birthday on. This means the second person will have a $\frac{364}{365} = 99.7\%$ chance of not sharing a birthday with anyone else.

Similarly, the third person can have his/her birthday on any of the remaining 363 days out of the year, which would give a $\frac{363}{365} = 99.5\%$ chance of the third person not sharing a birthday with the other two.

Using the multiplication rule, we can find the combined probability that all three people have a unique birthday.

$$Combined\ Probability = \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} = \frac{365 \times 364 \times 363}{365^3} \approx 99.2\%$$

So there is a 99.2% chance that 3 people in a room will *not* share a birthday.

If we extend this to 25 people, we will get:

$$CombinedProbability = \frac{365 \times 364 \times 363 \times 362 \times ... \times 344 \times 343 \times 342 \times 341}{365^{25}} \approx 43.1\%$$

This means there is a 43.1% chance that people will *not* share a birthday, or equivalently we can say there is a 56.9% chance that people *will* share a birthday.

This is the closest to 50%, so c) is the correct answer.

Scientific Equations

$100 Traveling at a speed of 30 kilometers per hour, how many minutes would it take to travel 1,500 meters?

The first thing to do on this question is to change all of our quantities into terms of the same units.

$$\frac{30\,kilometers}{1\,hour} = \frac{30\,kilometers}{60\,minutes} = \frac{0.5\,kilometers}{1\,minutes}$$

$$1,500\,meters = 1.5\,kilometers$$

Now that the units are sorted out, we can plug these quantities into the time equation.

$$Time = \frac{Distance}{Speed} = \frac{1.5\,kilometers}{0.5\,kilometers\,per\,minute} = 3\,minutes$$

$200 The speed of sound in air is about 343 meters per second. If you hear thunder 8 seconds after seeing a lightning bolt, approximately how far away was the bolt?

For this question we can use the distance equation:

$$Distance = Speed \times Time$$

Using the speed and the time given in the question, we can calculate a distance of:

$$Distance = Speed \times Time = 343\,\frac{meters}{second} \times 8\,seconds = 2,744 meters$$

So the lightning bolt was 2,744 meters, or 2.744 kilometers away.

(Note: This is not exactly correct, but it should be extremely close. If you wanted to be perfect, you could account for the time it would take for the light from the lightning bolt to reach your eyes as well!)

$300 One of Albert Einstein's famous equations is $E = mc^2$. From this, what equations can be used to solve for m and c?

$$m = \frac{E}{c^2}$$

$$c = \sqrt{\frac{E}{m}}$$

$400 If given voltage V, current I, resistance R and power P, what are two other *distinct* ways to write the power equation if $V = IR$ and $P = VI$?

To get another *distinct* equation, we can substitute the information from the voltage equation into the power equation.

Because we know that $V = IR$, we can write the V in the power equation as IR instead!

$$P = VI$$

$$P = (IR)I$$

$$P = I^2R$$

We can do similar steps to get a second equation except we will first have to find an equation for current.

$$V = IR$$

$$\frac{V}{R} = \frac{IR}{R}$$

$$\frac{V}{R} = I$$

Now we can write the I in the power equation as $\frac{V}{R}$.

$$P = VI$$

$$P = V(\frac{V}{R})$$

$$P = \frac{V^2}{R}$$

So the power equation can be written as:

$P = VI$ $\qquad$ $P = I^2R$ $\qquad$ $P = \frac{V^2}{R}$

$500 The Moon has a mass of about 7.35×10^{22} kilograms, and a radius of about 1,700 kilometers. What is its density in grams per cubic centimeter ($\frac{g}{cm^3}$)?

We want a density in grams per cubic centimeters, so first, lets change every unit in this question to be in terms of grams and centimeters.

$$7.35 \times 10^{22}\, kilograms \times 1,000\, \frac{grams}{kilogram} = 7.35 \times 10^{25}\, grams$$

$$1,700\, kilometers \times 1,000\, \frac{meters}{kilometer} \times 100\, \frac{centimeters}{meter} = 170,000,000\, centimeters$$

Next, we know that the density of any object can be found by using the density equation:

$$Density = \frac{Mass}{Volume}$$

We already know the mass, so now we have to find the volume. The Moon is in the shape of a sphere, so we need to find the volume of a sphere with a 170,000,000 centimeter radius.

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi \times 170,000,000^3 \approx 2.06 \times 10^{25}\, cm^3$$

Now we can find the density.

$$Density = \frac{Mass}{Volume} = \frac{7.35 \times 10^{25}\, grams}{2.06 \times 10^{25}\, cm^3} = 3.57\, \frac{grams}{cm^3}$$

Estimations

No calculators allowed!

\$100 Estimate the value of 209.75×3932.1625.

$$209.75 \times 3932.1625 \approx 200 \times 4,000 = 800,000$$

\$200 Estimate the value of $2^{300} \times \pi^{20}$.

Two approximations that we can use for this question at $2^{10} \approx 10^3$ and $\pi \approx \sqrt{10}$.

$$2^{300} \times \pi^{20} \approx 2^{10\times 30} \times (\sqrt{10})^{20}$$

$$\approx (2^{10})^{30} \times 10^{\frac{1}{2}\times 20}$$

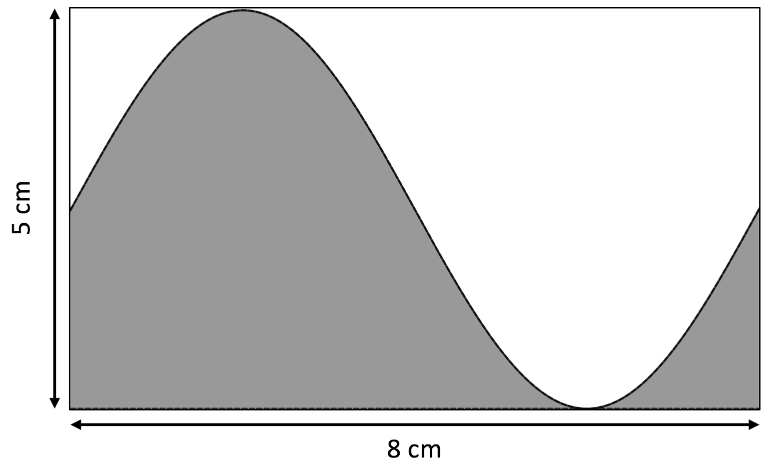
$$\approx (10^3)^{30} \times 10^{10}$$

$$\approx 10^{90} \times 10^{10}$$

$$\approx 10^{100}$$

\$300 Estimate the area of this shape:

The easiest way to do this estimation is to notice that the white area appears to be about the same as the gray area. This means that the area of the shape we're looking for (everything in gray), should approximately be half of the area of the entire rectangle.



$$Area \approx \frac{1}{2} \times Area\ Of\ Rectangle = \frac{1}{2} \times 5\ cm \times 8\ cm = 20\ cm^2$$

$400 Estimate how many centimeters the average fingernail grows per year.

You can base this calculation off of how often you cut your own fingernails, and how much length you have to cut off each time.

For example, if you cut your fingernails twice a month, and you cut off about a quarter of a centimeter each time, you would find that:

$$\frac{1}{4}\,\frac{centimeters}{cut} \times \frac{2\,cuts}{month} \times \frac{12\,months}{year} = 6\,centimeters\,per\,year$$

This is just one possible answer. The actual number is about 3.6 centimeters per year, so the estimation of 6 centimeters per year would be on the same order of magnitude, meaning it is a reasonable estimation!

$500 Estimate the value of:

$$\frac{(\pi^{2\sqrt{2}})^{\pi}}{\pi^2} + 2,500$$

$$\begin{aligned}
\frac{(\pi^{2\sqrt{2}})^{\pi}}{\pi^2} + 2,500 &\approx \frac{(3^{2\times 1.5})^3}{3^2} + 2,500 \\
&\approx 3^{(2\times 1.5\times 3)-2} + 2,500 \\
&\approx 3^7 + 2,500 \\
&\approx (3\times 3\times 3\times 3\times 3\times 3\times 3) + 2,500 \\
&\approx (9\times 9\times 9\times 3) + 2,500 \\
&\approx 3,000 + 2,500 \\
&\approx 5,500
\end{aligned}$$

The actual answer is about 5150.072.

The Scale of Numbers

$100 Sort these three numbers from smallest to largest: $2 \uparrow\uparrow 4$, 67 million, and 7.92×10^9.

$$2 \uparrow\uparrow 4 = 2^{2^{2^2}} = 2^{2^4} = 2^{16} = 65,536$$

$$67\ million = 67,000,000$$

$$7.92 \times 10^9 = 7,920,000,000$$

$200 Using the numbers 2, 5 and 7, what is the exponent tower that you can make with the largest value?

Generally, when trying to get the largest number when creating an exponent tower, you always want to have the largest numbers at the top of the exponent tower.

In this case, the biggest exponent tower that can be made is $2^{5^7} = 2^{78,125}$.

$300 Skewe's Number can be written as $S_k = 10^{10^{10^{34}}}$. What is the smallest n such that $2 \uparrow\uparrow\uparrow n$ is larger than S_k?

For this question, you can start at a value of $n = 1$ and work upwards until you achieve a number larger than S_k.

$$2 \uparrow\uparrow\uparrow 1 = 2$$

$$2 \uparrow\uparrow\uparrow 2 = 2 \uparrow\uparrow 2 = 2^2 = 4$$

$$2 \uparrow\uparrow\uparrow 3 = 2 \uparrow\uparrow (2 \uparrow\uparrow 2) = 2 \uparrow\uparrow 4 = 2^{2^{2^2}} = 65,536$$

$$\begin{aligned} 2 \uparrow\uparrow\uparrow 4 &= 2 \uparrow\uparrow (2 \uparrow\uparrow (2 \uparrow\uparrow 2)) \\ &= 2 \uparrow\uparrow (2 \uparrow\uparrow 4) \\ &= 2 \uparrow\uparrow 65,536 \\ &= 2^{2^{2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}}}} \end{aligned}$$

This number will be an exponent tower of twos made out of 65,536 twos! This will be much much bigger than Skewe's number, so the answer to this problem is n = 4.

$400 What is the value of n in the following equation:

$$(2 \uparrow\uparrow\uparrow 3) \times (2 \uparrow\uparrow 4) \times (4 \uparrow\uparrow 2) = 2^n$$

$$\begin{aligned}(2 \uparrow\uparrow\uparrow 3) \times (2 \uparrow\uparrow 4) \times (4 \uparrow\uparrow 2) &= (2 \uparrow\uparrow (2 \uparrow\uparrow 2)) \times (2^{2^{2^2}}) \times (4^4) \\ &= (2 \uparrow\uparrow 2^2) \times (2^{2^4}) \times (4 \times 4 \times 4 \times 4) \\ &= (2 \uparrow\uparrow 4) \times (2^{16}) \times (2^2 \times 2^2 \times 2^2 \times 2^2) \\ &= 2^{16} \times 2^{16} \times 2^2 \times 2^2 \times 2^2 \times 2^2 \\ &= 2^{16+16+2+2+2+2} \\ &= 2^{40}\end{aligned}$$

So n must be 40.

$500 What is the smallest magnitude number you can make with BEDMAS operations and the numbers: π, $\sqrt{2}$, 0.5, and 10 all at most once each?

The smallest magnitude number that I could find using these rules is:

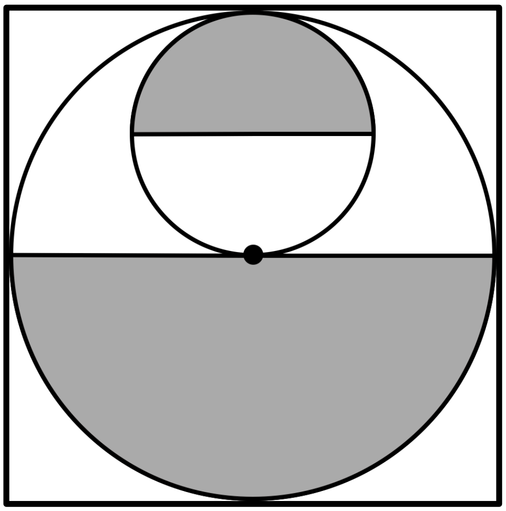
$$0.5^{10\times(\pi+\sqrt{2})} = 0.5^{45.56} = 1.93 \times 10^{-14} = 0.0000000000000193$$

However this might not be the smallest possible number that you can make. To check and see if it is, you would have to calculate every single different combination of numbers and operators (which would take an absurdly long time).

Final Jeopardy

Getting this final question correct will *double* your current score!

If the big circle in this diagram has an area of 50 cm^2, what is the area of the *un-shaded* region (i.e. everything that is white and inside the perimeter of the square)?



If the big circle has an area of 50 cm^2, the first thing we can do is find its radius.

$$Area = \pi \times R^2$$

$$50 = \pi \times R^2$$

$$\frac{50}{\pi} = \frac{\pi \times R^2}{\pi}$$

$$\sqrt{\frac{50}{\pi}} = \sqrt{R^2}$$

$$3.989\ cm = R$$

Now, because the circumference of the small circle lines up with the center of and the edge of the big circle, we can say that the radius of the big circle must be the diameter of the little circle. Alternatively we can say that the radius of the little circle must be half of the radius of the big circle.

$$R_{small\ circle} = R_{big\ circle} \div 2 = 3.989 \div 2 = 1.995\ cm$$

Now we can find the area of the small circle.

$$Area = \pi \times R^2$$

$$= \pi \times 1.995^2$$

$$= 12.5\,cm^2$$

The shaded part of the small circle will be half of its total area, or 6.25 cm^2.

The shaded part of the big circle will be half of its total area, or 25 cm^2.

This gives us a total shaded area of 31.25 cm^2.

To find the un-shaded area, we will have to take the area of the entire diagram, and subtract the area of the shaded regions. The entire diagram is a square (we can say for sure that it is a square because it perfectly encloses a circle), so we first need to find the side length of the square to find its area.

The side length of the square should be the same as the diameter of the big circle. The radius of the big circle is 3.989 cm, so if we double that, we find a diameter of 6.978 cm. The area of the square will then have to be:

$$Area = side^2$$

$$= 7.978^2$$

$$= 63.648\,cm^2$$

The area of the un-shaded region will then have to be:

$$Area_{Un-Shaded} = Area_{Total} - Area_{Shaded}$$

$$= 63.648\,cm^2 - 31.5\,cm^2$$

$$= 32.148\,cm^2$$