UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grades 7 & 8, Math Circles

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Series & Polygonal Numbers

Introduction

Mathematicians are always looking for easier and quicker ways to do mundane tasks. Adding up numbers is one such task. This week, we'll take a look at a tricky way to add consecutive natural numbers and then extend this idea to sums of both odd and even natural numbers.

Sum of the First n Natural Numbers

Say you were asked to find the sum of the natural numbers starting at 1 and going all the way to 10. Pretty easy right? You just write them down in a line and add!

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \underline{\qquad}$$

"Neat!", you say. However, one of your classmates thinks this was way too easy. He believes that *real* mathematicians can do this sum all the way up to fifty. "No problem, bring it on!", you say - you're no slouch! So you do the same thing as before and write out the numbers in a line so you can add them.

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + ...$$

Now we've got a slight problem - the page isn't wide enough for you to write out all the numbers! Even if it was, it'd take you a while to write them all out and add them. You're a mathematician. You don't have time for this!

Additionally, you classmate could've asked you to do the sum all the way up to 100 instead of stopping at 50. In fact, it's rumored that the famous mathematician **Carl Friedrich Gauss** (Figure 1.) was asked to do this sum by his teacher for unruly behavior in class.

Figure 1: Carl Friedrich Gauss

So how would you do it? You don't want to be adding numbers all day!

One way of thinking about this problem is to write out the series twice in the following way:

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & \dots & 96 & 97 & 98 & 99 & 100 \\ 100 & 99 & 98 & 97 & 96 & \dots & 5 & 4 & 3 & 2 & 1 \end{array}$$

Each column sums to 101

The three dots (or ellipsis) between 5 and 96 are there to tell the reader that you could keep writing out the numbers all the way from 5 to 96 but you're not going to.

Interestingly, all the columns give the same sum of 101. Since there are 100 columns, and each column sums to 101, the total sum of the two rows is _________.

You've found the sum of the two rows above. This is _________ the sum of the original series, which means that the sum of the original series is __________.

But how can you 'see' the answer?

When solving problems in math, it's always helpful to try and look for alternate ways to get the answer. Although you get the same answer, doing a problem in multiple ways helps you develop an intuition for how you get the answer. Additionally, you might find that one way is easier to remember or more intuitive than the other.

So then, how can you do the previous sum in an even more clever way?

One way of doing it is to think of tiny squares (unit squares). For simplicity, let's say each tiny square has an area of 1 (hence the name). What could tiny squares have to do with big sums?

Well, you can represent your sum using unit squares if you arrange them in a particular way.

Say for instance, you were asked to find the sum of all natural numbers from 1 to n (n just means that you could end the series wherever you like). So your series would look like:

$$1 + 2 + 3 + 4 + ... + (n-1) + n$$

We write the second last term as $(n-1)$ because we want the series to end at n. This means that adding 1 to the second last term should give us n. $(n-1)$ is our answer to the question: "What number comes before n?".

You could represent each term in the series with a corresponding number of unit squares and arrange them in a triangular pattern as shown in Figure 2 below. Again, the dots (or ellipsis) are there to indicate that you repeat the arrangement all the way to n rows.

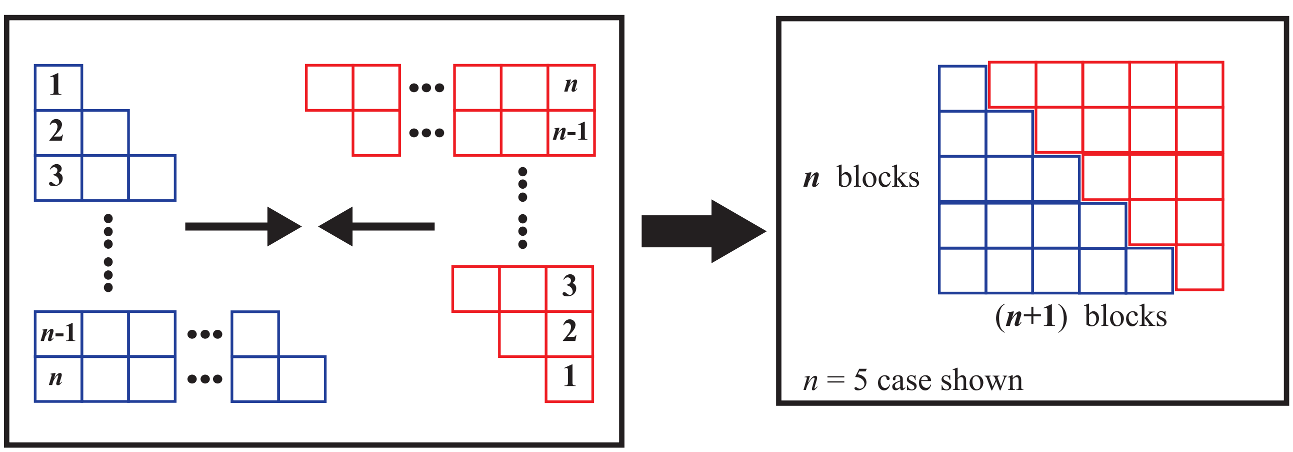

Figure 2: Representing the sum using unit squares

As seen in the figure, arranging two such 'triangles' leads to an interlocking rectangle with sides n and $(n+1)$. By summing the two series, we mean to count the total number of unit squares in the rectangle. So, the total number of unit squares in the rectangle in Figure 2 is __________.

In the formula we got above, notice that we didn't put a limit on how long the series could be, just that it had to end *somewhere*. This means that our formula applies for any series of consecutive natural numbers.

Another thing to remember is that n is the number of terms in the series, including the first and last terms. So a sum from 1 to 10 would have 10 terms, whereas a sum from 3 to 10 would have _______ terms.

Example 1:

For each of the following series, write down the number of terms (n) and the sum to n terms:

(a) $1 + 2 + 3 + 4 + ... + 31$

(b) $1 + 2 + 3 + 4 + ... + 92$

(c) $5 + 6 + 7 + 8 + ... + 24$

Hint: You know how to find the sum from 1 to n. Can you rewrite this series as a difference of two series that start at 1?

Sum of the First n Odd Natural Numbers

So now you've done a lot of thinking and come up with a couple of clever ways to add consecutive natural numbers really quickly. Your classmate is left speechless at the brilliance of your intellect.

You can now find the sum of any series of consecutive natural numbers. So what about consecutive **odd** or **even** numbers? Can you use something from the previous section that could help us find the sum of the first n odd numbers?

$$1 + 3 + 5 + 7 + 9 + 11 + ... + (2n - 1)$$

The $(2n-1)$ at the end of the above series is a general way of writing out any odd number. n is allowed to go from 1 to wherever you want your series to end. To see how this works, try plugging in $n = 1$ into it - you should get the first odd number. The formula allows you to answer the question: What's the n^{th} odd number?

Example 2:

What is:

(a) The 31^{st} odd number?

(b) The 50^{th} odd number?

(c) The 100^{th} odd number?

We were able to solve the previous problem visually, by arranging unit squares into triangles and then finding the total number of unit squares in an interlocking rectangle made up of two such triangles. We could do something similar here too.

As before, we represent each term in the series with the corresponding number of unit squares. Interestingly, this time we're able to arrange the unit squares into a larger square (as shown in Figure 3).

Weirdly, the area of each *big* square is always a perfect square number. For example, the first *big* square has an area of $1 + 3 = 4$ which means that the sum of the first 2 odd numbers is 4. Similarly, the second *big* square, which represents the sum of the first 3 odd numbers has an area of ____________.

So if you had to find the sum of the first n odd numbers, your sum would be (in terms of n) ____.

That's interesting! The sum of the first n consecutive odd numbers is always a perfect square number!

Another way of thinking about this is to subtract two consecutive square numbers (say 4 and 9). You always end up with an odd number! In other words, the *spacing* between any two consecutive perfect square numbers, is an odd number. What this means is that the square numbers alternate between odd and even numbers, just like the natural numbers!

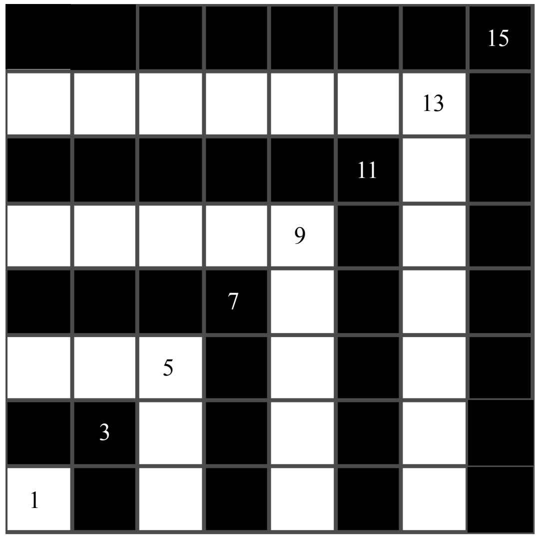

Figure 3: Sum of odd numbers represented using unit squares

Example 3: Find the sum of these series:

(a) $1 + 2 + 3 + 4 + 5 + ...$ to 22 terms.

(b) $3 + 6 + 9 + 12 + 15 + 18 + ...$ to 22 terms.

Hint: This isn't as hard as it looks. See how the terms are related to each other?

(c) $2 + 6 + 10 + 14 + 18 + 22 + 26 + ...$ to 17 terms.

Hint: Same trick as in part (b). How can you get each term from the previous one?

Sum of the First n Even Natural Numbers

Now that we've looked at consecutive natural numbers and odd numbers, the only thing left is the even numbers. The series we're looking at is:

$$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + ... + 2n$$

Similar to what we did with the odd numbers, the last term here is $2n$ and n is allowed to go from **1** all the way to whatever you like. $2n$ is how you'd represent any even number.

Try plugging in different values for n like last time. $n = 1$ should give you the first even number.

Now that you've handled two different series by arranging unit squares, how would you arrange them this time? *(see Example 4(b))*

If you look at the series a bit closer, you can find a general way of writing the sum to n terms. See how each term in the series is the previous one plus 2? This means that if you wanted the 4^{th} term of the series, you'd start with 2 (the first term) and add 2 three times, giving you _______________.

What do we call repeated addition? Multiplication of course! How can you use this fact to solve our current problem?

Example 4:

(a) Find the sum of the series: $2 + 4 + 6 + 8 + 10 + ... + 50$

(b) Find the sum of the series: $2 + 4 + 6 + 8 + ... + 2n$

Hint: Remember our trick from Example 3?

(c) **Bonus:** Find a way to arrange unit squares so that you get the same answer as in part (b).

Polygonal Numbers

As you've probably noticed, the title of this week's lesson is called **Series & Polygonal Numbers**. I know what you're thinking: "Where's the polygonal numbers bit?"

Notice how both the previous problems were solved in a pretty neat way by arranging unit squares into familiar shapes (triangles and squares). You could've used circles instead of unit squares too. In fact, it doesn't matter what object you use to represent each term in the series. The only rule is that the objects need to be identical.

Take Figure 4 (next page) for instance. We already know that it represents the sum of the first n natural numbers - so the leftmost triangle represents the sum of the first natural number i.e 1, the next triangle represents the sum of the first two natural numbers i.e, $1 + 2 = 3$ etc. Notice how each *layer* of the triangle represents each term in the series (they're shaded differently for this reason). So say you wanted to find the sum of the first 3 natural numbers ($n = 3$). You would count the number of circles in the first 3 layers starting from the top.

Each of these sums i.e. $1, 3, 6, 10, ...$ is called a **triangular number** because you represent them by arranging identical things in equilateral triangles. T_n is the symbol we use when we want to talk about the n^{th} triangular number.

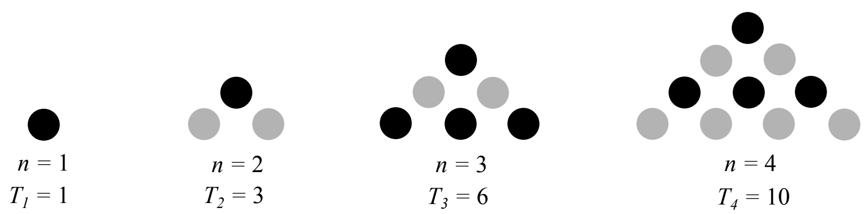

Figure 4: The triangular numbers

We already know how to find the n^{th} triangular number. Remember, the triangular numbers represent the sum of consecutive natural numbers (starting at 1). So the formula for the n^{th} triangular number (T_n) must be: $T_n =$ ____________.

Similarly, Figure 5 represents the sum of the first n odd natural numbers. So the second figure from the left represents the sum of the first two odd numbers $1 + 3 = 4$. The *layers* in this case are shaded as before. Each new layer represents adding a new term to our sum (just as we did before).

In this case, each sum i.e. $1, 4, 9, 16, 25, 36, ...$ is called a **square number**. You probably already know this because square numbers pop up everywhere in geometry thanks to a particular old Greek man (Pythagoras). S_n is the symbol we use when we want to talk about the n^{th} square number.

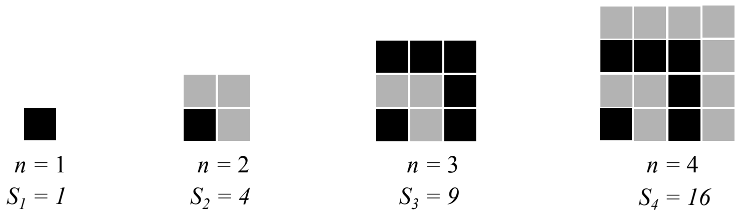

Figure 5: The square numbers

You already know how to find the square of any given number. This means that the formula for the n^{th} square number is: $S_n =$ ________.

You could go even higher with **pentagonal** and **hexagonal** numbers, but they're a bit trickier to handle. In general, these numbers are called **polygonal numbers**. They're the number of dots/squares/identical things needed so that they can be arranged into a **regular polygon**.

Now you might've noticed that I've left out the sum of the first n even numbers. That's because these sums aren't really polygonal numbers, as you've probably noticed from Example 3.

You can use polygonal numbers to solve all sorts of neat problems. One of them is particularly famous and is called **The Handshake Problem**. It's one of the problems on this week's problem set.

Problem Set

1. Find the sum of all the natural numbers between 17 and 31.

2. Find the sum of all the odd numbers between 100 and 200.

3. (a) Find the 11^{th} square number

   (b) Find the 9^{th} triangular number

   (c) Dividing a particular square number by 3 gives 108. What's the square number?

   (d) Squaring a particular triangular number and adding 10 to the result gives 46. What is the triangular number?

   Bonus: Write your result as T_n, where n is the position of your triangular number in the list of triangular numbers.

   Hint: You'll need to find n *to answer the bonus part*

4. Find the sum of the following series:

$$\tfrac{7}{13} + \tfrac{14}{13} + \tfrac{21}{13} + \tfrac{28}{13} + \ldots$$

   to 20 terms.

   Hint: See Examples 3 and 4.

5. You're delivering newspapers as part of your summer job and you're paid at $31 a week. Your supervisor is so impressed with your speed that she increases your weekly pay by $2, for each of the remaining 12 weeks. How much money do you have at the end of the summer if you spend $100 a month on ice cream and save the rest?

6. A group of n mathematicians are at a conference. How many distinct handshakes occur assuming that they're all polite and never turn down a handshake?

   Hint: Imagine that each person goes round handshaking everyone else. How many handshakes would the first person give? What about the second person?...