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Grades 7 & 8, Math Circles

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Series & Polygonal Numbers

Solutions

Example 1

(a) $1 + 2 + 3 + 4 + ... + 31$

Let's call the sum of this series R.

$R = 1 + 2 + 3 + 4 + ... + 31$

We know that the sum S of the consecutive natural numbers from 1 to n is:

$S = \frac{n(n+1)}{2}$

In other words, the sum S is:

$S = \frac{\textit{(Number of Terms in Series)(First term + Last Term)}}{2}$

There are 31 terms in our series and the sum of the first and last terms is $31 + 1 = 32$.

This means:

$R = 1 + 2 + 3 + 4 + ... + 31 = \frac{31 \times 32}{2} = \underline{\mathbf{496}}$

(b) $1 + 2 + 3 + 4 + ... + 92$

Again, let's call the sum of this series R.

There are 92 terms in our series and the sum of the first and last terms is $1 + 92 = 93$.

This means:

$R = 1 + 2 + 3 + 4 + ... + 92 = \frac{92 \times 93}{2} = \underline{\mathbf{4278}}$

(c) $5 + 6 + 7 + 8 + ... + 24$

Let's call the sum of this series P for a change. Notice how the series starts at 5 instead of 1. We can still use our general formula from part (a) to find the sum.

There are $24 - 5 + 1 = 20$ terms in our series. We add one at the end because we want to include the first and last terms in our count.

The sum of the first and the last terms is $5 + 24 = 29$.

So, the sum of our series is:

$P = 5 + 6 + 7 + 8 + ... + 24 = \frac{20 \times 29}{2} = \underline{\mathbf{290}}$

Example 2

(a) To find the n^{th} odd number, we can use the formula $(2n - 1)$.

So the 31^{st} odd number is:

$(2 \times 31 - 1) = 62 - 1 = \underline{\mathbf{61}}$

(b) Similarly, the $50th$ odd number is:

$(2 \times 50 - 1) = 100 - 1 = \underline{\mathbf{99}}$

(c) Finally, the 100^{th} odd number is:

$(2 \times 100 - 1) = 200 - 1 = \underline{\mathbf{199}}$

Example 3

(a) $1 + 2 + 3 + 4 + ...$ to 22 terms

Let's call the sum of this series R.

Here you're given the number of terms i.e. 22. Now all you need is the sum of the first and last terms of the series. You know that the first term is 1. The last term must be 22 (since your finding the sum to 22 terms). This means that the sum of the first and last terms is $1 + 22 = 23$.

So:

$R = 1 + 2 + 3 + 4 + ...$ to 22 terms $= \frac{22 \times 23}{2} = \underline{\mathbf{253}}$

(b) $3 + 6 + 9 + 12 + 15 + 18 + ...$ to 22 terms.

Let's call the sum of this series S.

Notice how this series is different from the ones we've seen before. It starts at 3 and all it's terms are consecutive multiples of 3.

In fact, we can rewrite our series in the following way:

$S = (3 \times 1) + (3 \times 2) + (3 \times 3) + (3 \times 4) + (3 \times 5) + ...$ to 22 terms.

Since there is a common factor of 3 everywhere in our series, we can pull it out and rewrite the series:

$S = 3 \times (1 + 2 + 3 + 4 + 5 + ...$ to 22 terms$)$

We already found the sum of the series in the brackets! From part (a) the sum of that series is $\underline{\mathbf{253}}$.

This means that S must be 3 times the sum we found in part (a).

So:

$S = 3 \times 253 = \underline{\mathbf{759}}$

(c) $2 + 6 + 10 + 14 + 18 + 22 + 26 + ...$ to 17 terms.

Let's call the sum of this series Q.

This series is weird too - it starts at 2 and jumps by 4. However, notice that all the terms of the series are even numbers. All even numbers are multiples of 2. So we can actually rewrite this series as:

$Q = (2 \times 1) + (2 \times 3) + (2 \times 5) + (2 \times 7) + (2 \times 9) + (2 \times 11) + ...$ to 17 terms.

Since you've got a common factor of 2 everywhere, you can pull it out like we did in part (b). This gives you:

$Q = 2 \times (1 + 3 + 5 + 7 + 9 + 11 + ...$to 17 terms$)$

We already know the sum of the series in the brackets - it's the sum of the first n odd numbers! The sum of the first n odd numbers has the formula n^2. Since our series stops at 17 terms, our sum will be:

$17 \times 17 = 17^2 = \underline{\mathbf{289}}$

So the sum of our series is just 2 times this number:

$Q = 2 \times 289 = \underline{\mathbf{578}}$

Example 4

(a) $2 + 4 + 6 + 8 + 10 + ... + 50$.

Let's call the sum of this series S. We can use the same trick we used in parts (b) and (c) of Example 3 (pulling out a common factor). Every term in this series is even and so is a multiple of 2, which means that we can pull out a common factor of 2. First we rewrite the series as:

$S = (2 \times 1) + (2 \times 2) + (2 \times 3) + (2 \times 4) + ...(2 \times 25)$

Then we pull out the common factor of 2:

$S = 2 \times (1 + 2 + 3 + 4 + 5 + ...25)$

Again, we see a familiar series inside the brackets. There are 25 terms in the series inside the brackets and the sum of the first and last terms is $1 + 25 = 26$.

So the sum of the series in the brackets is: $\frac{25\times26}{2} = \underline{\mathbf{325}}$

So the sum of our series S, is just 2 times 325:

$S = 2 \times 325 = \underline{\mathbf{650}}$

(b) $2 + 4 + 6 + 8 + 10 + ... + 2n$

Let's call R the sum of this series.

This series is exactly the same as the one we saw in part (a), except that we allow it to stop at *any* even number (which we represent by $2n$).

We do the same trick as before and pull out a common factor of 2 and rewrite the series as:

$R = 2 \times (1 + 2 + 3 + 4 + 5 + ... + n)$

The sum of the series inside the bracket is the sum of the first n consecutive natural numbers. The sum of the first n consecutive natural numbers is given by the formula $\frac{n(n+1)}{2}$.

So our series, which is the sum of the first $2n$ *even* numbers, must be 2 times this:

$R = 2 \times \frac{n(n+1)}{2} = \underline{\mathbf{n(n+1)}}$

(c) You can arrange the unit squares in exactly the same way as you did for the sum of the first n consecutive natural numbers (Figure 1). They key difference here is that you don't divide by two - your sum would just be the number of unit squares in the interlocking rectangle (right) i.e. $\underline{\mathbf{n(n+1)}}$.

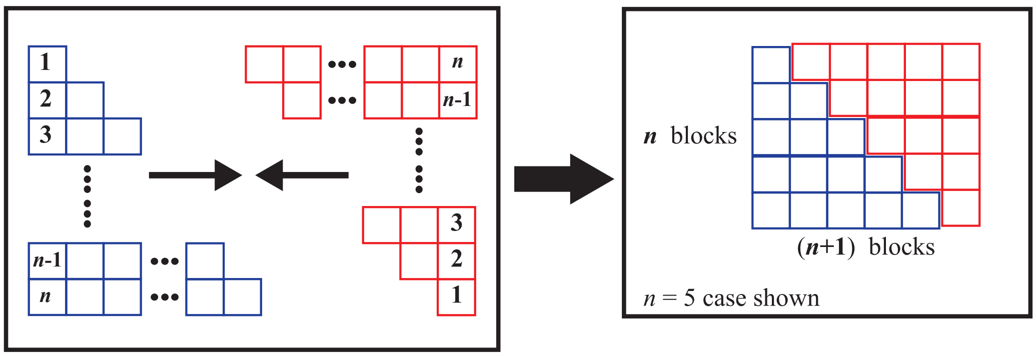

Figure 1: Arranging unit squares to find the sum of the first $2n$ even numbers

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Problem Set

1. The question asks you to find the sum of all the natural numbers between 17 and 31 (including both 17 and 31).

   So the sum you need to find is:

   $S = 17 + 18 + 19 + 20 + 21... + 31$

   We can use our formula from Example 1 (a) to solve this problem.

   The sum of any series of consecutive natural numbers is:

   $\frac{\textit{(Number of Terms in Series)(First term + Last Term)}}{2}$

   Here the number of terms is $31 - 17 + 1 = 15$ and the sum of the first and last terms is $17 + 31 = 48$.

   So our sum is:

   $S = \frac{15 \times 48}{2} = \underline{\mathbf{360}}$

2. This time the question asks you to find the sum of all the odd numbers between 100 and 200 (but this time we've got to exclude 100 and 200 because they're not odd numbers).

So the sum you need to find is:

$S = 101 + 103 + 105 + ... + 199$

Using the same formula as in Question 1, we see that the number of terms in our series is just the number of odd numbers between 100 and 200, which is just 50 (since every other number between 100 and 200 is odd). The sum of the first and last terms is $101 + 199 = 300$.

So our sum is:

$S = \frac{50\times300}{2} = \underline{\mathbf{7500}}$

3. (a) The formula for the n^{th} square number is simply $S_n = n^2$ (where we write S_n to mean the n^{th} square number). This means that the 11^{th} square number S_{11} is:

   $S_{11} = 11^2 = \underline{\mathbf{121}}$

   (b) The formula for the n^{th} triangular number is:

   $T_n = \frac{n(n+1)}{2}$

   We write T_n to mean the n^{th} triangular number. This means that the 9^{th} triangular number is:

   $T_9 = \frac{9(9+1)}{2} = \frac{9\times10}{2} = \underline{\mathbf{45}}$

   (c) Let's call our unknown square number s. Our question says that if we divide s by 3, we get 108. We can write this out as an equation in terms of s:

   $\frac{s}{3} = 108$

   We can rearrange this equation to get:

   $s = 108 \times 3 = \underline{\mathbf{324}}$

   So our square number is 324. Notice that $324 = 18^2$. So our answer is actually the 18^{th} square number.

   (d) Let's call our unknown triangular number t. The question says that if we square t and add 10 to the result, we get 46. Translating this into math we get:

   $t^2 + 10 = 46$

   We can rearrange this equation to get:

   $t^2 = 36$

   So this equation tells us that if we multiply our unknown triangular number with itself, we get 36. We know that $6 \times 6 = 36$, which means that 6 must be our triangular number.

   $\therefore t = \underline{\mathbf{6}}$ (The three dots mean 'therefore')

Bonus:

We need to write our triangular number in the form T_n, where n is the position of our triangular number (i.e. 6), in the list of triangular numbers.

Remember, the formula for the n^{th} triangular number is:

$T_n = \frac{n(n+1)}{2}$

So we plug our triangular number into the right side of this equation, giving:

$6 = \frac{n(n+1)}{2}$

This equation tells us that half the product of two consecutive numbers (i.e. n and $(n+1)$) gives us six. Rearranging this gives:

$12 = n(n+1)$

Now this equation tells us that the product of two consecutive numbers must give us 12. 3 and 4 are two such numbers such that $3 \times 4 = 12$. So n must be $\underline{\mathbf{3}}$. I.e.:

$n = \underline{\mathbf{3}}$

So 6 is the 3^{rd} triangular number. In other words:

$T_3 = 6$

4. Let's call the sum of our series S.

   $S = \frac{7}{13} + \frac{14}{13} + \frac{21}{13} + \frac{28}{13} + ...$ to 20 terms.

   Remember our trick in Examples 3 and 4 where we pulled out a common factor? We do the same thing here. The difference is that instead of being a nice number like 2 or 3, the common factor here is actually $\frac{7}{13}$. To see this, notice that the denominators of all the terms in the series is the same i.e. 13. Also see how the numerators are all multiples of 7?

   So, we can rewrite our series like this:

   $S = (\frac{7}{13} \times 1) + (\frac{7}{13} \times 2) + (\frac{7}{13} \times 3) + ...$ to 20 terms.

   We can pull out the common factor of $\frac{7}{13}$ and rewrite the series like this:

   $S = \frac{7}{13} \times (1 + 2 + 3 + 4 + 5 + ...$to 20 terms$)$

   Again, we already know how to find the sum of the series in the brackets. There are 20 terms and the sum of the first and the last terms is $1 + 20 = 21$. So the sum of the series in the brackets is:

   $\frac{20 \times 21}{2} = \underline{\mathbf{210}}$

   This means that our sum S is $\frac{7}{13}$ times 210.

   $\therefore S = \frac{7}{13} \times 210 = \frac{1470}{13}$

5. This problem is an example of a word problem that requires you to find the sum of a series to solve it.

   You know that you're paid at \$31 for the first week. Your supervisor is impressed by your work so she increases this to \$33 for the second week, \$35 for the third week and so on. Each time the weekly pay increase is by \$2.

   Let's call E your total earnings for the summer. We can write E as the sum of your weekly earnings for each of the 13 weeks:

   $E = \$31 + \$33 + \$35 + \$37 + \$39 + ...$ to 13 terms

   This looks similar to the odd number series we solved earlier. We can rewrite E to make this more obvious:

   $E = (30 + 1) + (30 + 3) + (30 + 5) + ...$ to 13 terms

   Notice how the 30 appears in all the terms in brackets? In fact, 30 appears 13 times. We can use this fact to rewrite E as:

   $E = (30 \times 13) + (1 + 3 + 5 + 7 + ...$ to 13 terms)

   We already know the sum of any odd number series that starts at 1 - it's just the number of terms squared. So the sum $1 + 3 + 5 + 7 + ...$ to 13 terms is just $13^2 = \underline{\mathbf{169}}$.

   This means that your total earnings for the summer is:

   $E = (30 \times 13) + (169) = \underline{\mathbf{559}}$

   However, this isn't your final answer. You're given that you spend \$100 on ice cream each month. One month has 4 weeks on average. This means that 13 weeks is $13/4 = \underline{\mathbf{3.25}}$ months.

   So over 13 weeks, you spend $3.25 \times 100 = \underline{\mathbf{\$325}}$ on ice cream.

   So the total amount you're left with at the end of the summer is:

   $\textit{Savings at the end of summer} = \textit{Total Earnings} - \textit{Total amount spent on ice cream}$

   In other words:

   $\boldsymbol{\textit{Savings at the end of summer}} = \$559 - \$325 = \underline{\mathbf{\$234}}$

6. This is an example of a problem in which the triangular numbers arise naturally.

   Imagine the mathematicians shaking hands with each other, one by one. We start by noticing that the first mathematician of the group makes $(n - 1)$ unique handshakes with everyone else. He/she doesn't shake hands with himself, which is why we have $(n - 1)$ and not n.

Similarly, the second mathematician shakes hands with everyone except the first mathematician and himself/herself. This means that the second mathematician makes $(n-2)$ unique handshakes.

Following the same line of reasoning, the third mathematician makes $(n-3)$ unique handshakes and so on. However, the n^{th} mathematician makes no unique handshakes.

We can see this easily if we consider the case of just 3 mathematicians A, B and C (i.e. $n=3$). A shakes hands with both B and C. So we get 2 unique handshakes.

B can shake hands with only C as A and B have already shook hands. So we have 3 unique handshakes in total.

Finally, C can't shake hands with both A and B because both A and C and B and C have shook hands earlier. So the third mathematician in the group (i.e. C), makes no unique handshakes.

Let's call H the total number of handshakes in a group of n mathematicians. By following our reasoning from earlier, we can write H as the sum of unique handshakes made by each mathematician:

$H =$ *(# of unique handshakes made by the* 1^{st} *mathematician)* $+$
(# of unique handshakes made by the 2^{nd} *mathematician)* $+$
(# of unique handshakes made by the 3^{rd} *mathematician)* $+ \ldots +$
(# of unique handshakes made by the n^{th} *mathematician)*

In other words:

$H = (n-1) + (n-2) + (n-3) + \ldots + 1 + 0$

This is just the sum of the consecutive natural numbers from 1 to $(n-1)$. We can use our formula from Example 1 to find this sum:

$H = \frac{\textit{(Number of Terms in Series)(First term + Last Term)}}{2} = \frac{(n-1)\times(n-1+1)}{2} = \frac{(n-1)n}{2}$

This is because there are $(n-1)$ terms in our series and the sum of the first and last terms is $(n-1+1) = n$.

Interestingly, our formula for H is exactly the same as for the $(n-1)^{th}$ triangular number T_{n-1}.

Variations of the handshake problem are studied in computer networks and the branch of mathematics that studies interconnected things is called *Graph Theory*.