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Grades 7 & 8, Math Circles

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Angles & Circles

Introduction

Circles are an important topic in math. You come across them every single day. This week, we're going to look at a couple of interesting facts about circles and their relation to how we measure angles. We'll also learn how to measure angles in units other than degrees.

Circumference

Imagine that you're walking along the edge of a circle which has a radius of 1 unit. You walk all the way around the circle until you get back to the exact spot where you started. The distance you've traveled is called the *circumference*.

You probably know how to calculate this distance from the formula:

$$\textit{Circumference} = 2\pi R$$

where R is the radius of the circle you're interested in and π is a special number that is given to you (usually as approximately 3.14 or $\frac{22}{7}$).

The *circumference* is to circles what the *perimeter* is to triangles, rectangles, squares and other polygons.

Example 1:

(a) Find the circumference of a circle with a radius of:

(i) 1 cm

(ii) 5 m

(iii) 0.5 cm

(b) Find the circumference of a circle with a diameter of:

(i) 1 cm

(ii) 0.5 cm

(iii) 10 m

(iv) d *units*

So where did this mysterious 2π in the circumference formula come from? Read on to find out.

Measuring Angles In Degrees

You probably already know what angles are. It's the figure you get when two lines intersect. You also know that we measure angles in degrees using a **protractor** which is like a ruler for angles and has equally spaced markings on it.

You know that a circle is divided into 360 degrees (°). But why 360°? Why not 400° or 1762°? People think that this is primarily due to the ancient Sumarians, who discovered that there were about 360 days in a year. This wasn't too far off from what we now know to be 365 days, so the ancient Sumarians were smart people! We've stuck with this convention ever since.

Actually, there's no reason why you can't have only 4° in a circle (Figure 1). In fact, you could *define* that a circle is made up of only 4° instead of the usual 360°. In this case, perpendicular lines would intersect at 1° instead of your usual 90°.

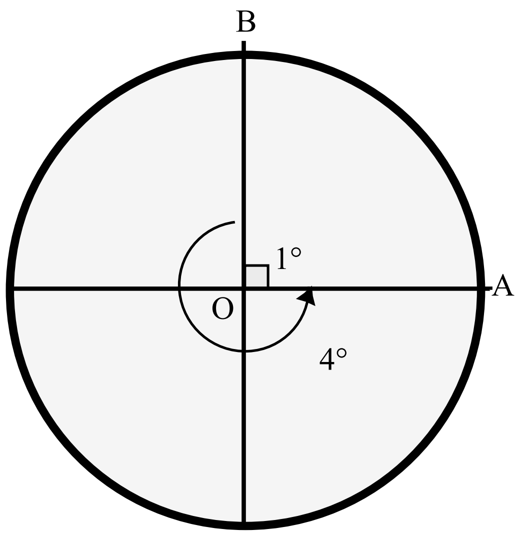

Figure 1: Two lines intersect at 1° if we define a full circle to have 4°

In Figure 1, notice that the lines are still perpendicular. What's changed is our measurement of the angle between them. Changing how we choose to measure an angle shouldn't change the actual angle itself. It's sort of like how pineapples are called different names in different languages - this doesn't change the fact that they're still pineapples!

When you turn by 5°, it means that you're turning around by $\frac{5}{360}^{th}$s of a complete turn. Saying that you're turning by $\frac{90}{360}^{th}$s of a complete turn or saying that you're turning by a quarter of a complete turn, mean exactly the same thing. This is why the choice of 360° is arbitrary - it can be whatever you want!

Example 2:

How would you write 90°, 45°, 180° and 0° if a circle had:

(a) 40°

(b) 100°

(c) 180°

instead of the usual 360°?

Arc Length

Imagine an ant (let's call him Sebastian), walking along the edge of a circle of radius R centered at O (Figure 2).

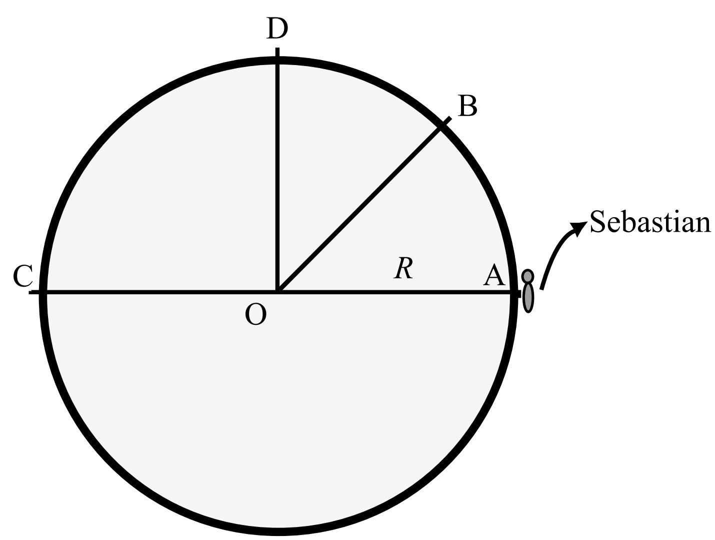

Figure 2: Sebastian the ant

Sebastian starts walking at point A and reaches point B. The distance that Sebastian walked is the length of the *arc* $\widehat{\text{AB}}$. How would you find this length?

Well, you already know that if Sebastian had walked all the way round the circle such that he ended up back at point A, he'd have walked a distance that's equal to the circumference of the circle (i.e. $2\pi R$).

If Sebastian had walked only *halfway* round the circle such that he stopped at point C (i.e. along arc $\widehat{\text{AC}}$), he'd have walked *half* the circumference of the circle (i.e. $\widehat{\text{AC}} = \frac{2\pi R}{2} = \pi R$).

Similarly, if Sebastian had walked only a quarter of the way round (along arc $\widehat{\text{AD}}$), he'd have walked a *quarter* of the circumference of the circle (i.e. $\widehat{\text{AD}} = \frac{2\pi R}{4} = \frac{\pi}{2}R$).

See a pattern?

The distance that Sebastian travels on any arc is a fraction of the circumference of the circle. Translating this into math, we write:

Distance traveled by Sebastian ***(Arc Length)***

= *(Circumference of circle)* × *(Fraction of the circle that Sebastian covers)*

Example 3:

Sebastian the ant is walking along a circle of radius 4 units. Find the distance (arc length) he travels, if he covers:

(a) $\frac{1}{8}^{th}$ of the entire circle.

(b) $\frac{3}{4}^{th}$ of the entire circle.

(c) $\frac{2}{3}^{rd}$ of the entire circle.

Now we can calculate the distance Sebastian travels, if we're given the radius of the circle he's on and the fraction of the circle he covers. But what if we're given an angle instead? (Figure 3)

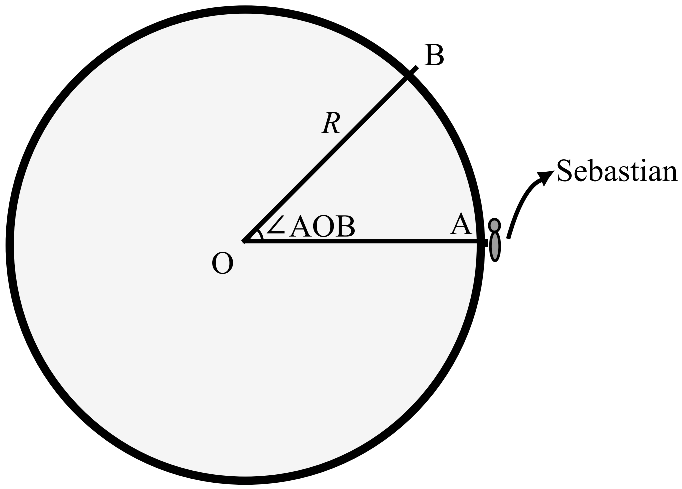

Figure 3: Finding the distance Sebastian travels when the angle is given

How can we get the fraction of the circle Sebastian covers, if we're given the angle $\angle AOB$?

Remember how $1°$ is actually $\frac{1}{360}^{th}$ of a full turn? How about $90°$? $90°$ is actually $\frac{90}{360}^{th} = \frac{1}{4}^{th}$ of a full turn. So we already know how to convert degrees to fractions of a turn!

This means that the fraction of the circle that Sebastian covers is:

$$Fraction\ of\ the\ circle\ that\ Sebastian\ covers = \frac{\angle AOB(in\ degrees)}{360°}$$

Now we can write down a formula for finding the arc length when we're given only the radius and the angle made (or *subtended*) by the arc at the center of the circle.

So our arc length formula is:

$$Arc\ Length = \frac{Angle\ subtended\ by\ arc\ at\ the\ center}{360°} \times Circumference$$

Example 4:

Find the arc length:

(a) When the arc subtends an angle of $45°$ at the center of a circle of radius 4 units.

(b) When the arc subtends an angle of $30°$ at the center of a circle of radius π units.

Radians

So now we know how to calculate the length of any arc given the angle subtended by it at the center. But there's a problem with the arc length formula that we used in the previous section - it depends on how many degrees we define to be in a circle. In other words, if someone came along and defined that a circle was made up of $400°$ instead of the usual $360°$ and asked us to calculate the arc length, we'd have to change our formula. If instead, they defined that a circle had $1072°$, we'd have to change the formula once again.

What if we rewrote our formula for the arc length so that it doesn't depend on how many degrees other people think there are in a circle? It'd save us quite a bit of time!

First, let's rewrite our formula like this:

$$L = \frac{x}{360°} 2\pi R$$

To make this easier to write, we've assigned the following symbols:

Arc Length $\rightarrow L$

Circumference $\rightarrow 2\pi R$

Angle subtended by the arc at the center (in degrees) $\rightarrow x$

We don't want our arc length formula to depend on how many degrees there are in a circle. So, we group the 2π and the $\frac{x}{360°}$ together and call the product θ (Theta). In math, this means:

$$\theta = 2\pi \frac{x}{360°}$$

So now, our arc length formula looks like this:

$$L = R\theta$$

See how it looks much simpler? So now, if someone gives us an angle in degrees, we ask them to divide it by 360° and multiply it by 2π so that we don't have to change our formula.

So now we're measuring angles as fractions of 2π. "Well how's this easier? It's more complicated!", you say. Although it might initially seem complicated, see how our formula became very simple? Often when you're doing harder math, it's easier if we change our units so that our formulas and equations look less cluttered.

This new angle measure is called a **radian**. It's called 'radian', because our new angle measure tells you how much you've moved along the edge of a circle in terms of the radius of the circle. For example, if you moved a distance of R along the edge of a circle of radius R (Figure 4), you'd have moved by **1 *radian*** (or 1 *rad.*).

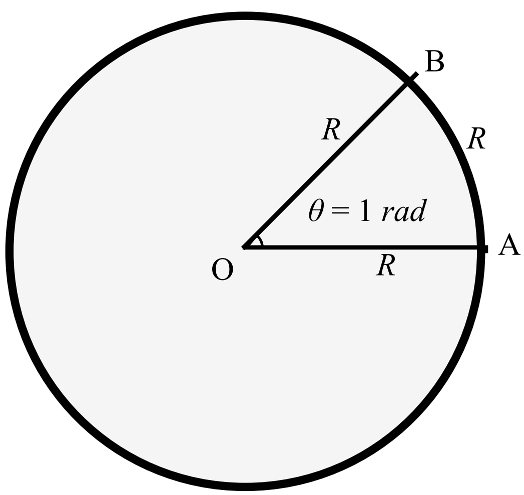

Figure 4: The definition of 1 radian

Similarly, if you moved a distance of $4a$ along the edge of a circle of radius a, you would've moved **4 *radians***.

Changing Between Degrees & Radians

Remember how we clumped together the angle (in degrees) and the 2π into a new symbol θ? We can use that definition to convert degrees to radians (and vice-versa).

We defined θ as:

$$\theta(radians) = 2\pi\tfrac{x(degrees)}{360°}$$

We can cancel out the factor of two and rewrite this as:

$$\theta(radians) = \pi\tfrac{x(degrees)}{180°}$$

This formula tells you how to get an angle in radians (θ), if you start out with it in degrees (x).

To go the other way, we can rearrange the equation to get:

$$x(degrees) = \tfrac{180°}{\pi}\theta(radians)$$

Example 5:

(a) Convert the following angles from degrees to radians:

- (i) $1°$
- (ii) $360°$
- (iii) $90°$
- (iv) $180°$

(b) Convert the following angles from radians to degrees:

- (i) 1
- (ii) π
- (iii) 2π
- (iv) $\frac{\pi}{3}$

So let's step back a bit and see how measuring angles in radians differs from measuring them in degrees.

Think of Sebastian the ant, walking along the edge of his circle. A degree measure tells you how much you need to turn your head by to keep him in your sights. A radian measure on the other hand, tells you how far Sebastian walked along the edge of the circle, in terms of it's radius. However, both degrees and radians measure the *same* angle - it's just that they do it in different ways.

It's important to understand that what we learned so far does not mean that degrees are useless - they're still used in a lot of places like car (automotive) design, geography and even rocket science.

Radians are generally used by mathematicians when they want to express mathematical facts about angles in some *natural* units because the math becomes more straightforward (as you've already seen).

Problem Set

1. Find the circumference of the circle if:
   (a) The radius is 3 units.
   (b) The radius is 2.5 units
   (c) The diameter is 4 units
   (d) The diameter is 2π units.
2. Find the arc length if:
   (a) The radius is 2 units and the angle subtended by the arc at the center is $60°$.
   (b) The diameter is 2 units and the angle subtended by the arc at the center is 1 radian.
3. Which figure has a greater perimeter?
   (a) A square of side 4 units.
   (b) A circle of radius 4 units.
   (c) An equilateral triangle of side 4 units.

4. Which figure encloses the biggest area if the perimeter is fixed to 12 units?

   (a) A circle.

   (b) A square.

   (c) An equilateral triangle.

   Hint: Area of triangle $= \frac{1}{2} \times (base) \times (height)$, *Area of a square* $= (side)^2$, *Area of a circle* $= \pi(radius)^2$

   Hint: $(base)^2 + (height)^2 = (hypotenuse)^2$ *(Pythagoras' Theorem for part (c)).*

5. Jenson has a mountain bike which has 20" wheels. Unfortunately, he forgot to bolt on his front wheel completely before moving off. As a result, the wheel falls off and keeps rolling down the street until it comes to a stop 100 feet away. How many complete rotations does his front wheel make before it stops rolling?

6. Farmer John has a circular field where he grazes his cows to make his famous *Farmer John Cheese*. Within this field, he has a circular pen where he keeps his sheep. If the pen has a diameter of 10 m and the whole field has a diameter of 30 m, what is the total grazing area available to the cows?

   Hint: The area of a circle of radius r *is* πr^2.