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Mathematics and Computing

Grades 7 & 8, Math Circles

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Series & Polygonal Numbers

Solutions

Example 1

(a) (i) We're given that the radius of the circle we're interested in is 1 cm. This means that we can write our circumference as:

$$Circumference = 2\pi R = 2\pi \times 1\ cm = 2\pi\ cm$$

Notice that we could have chosen to plug in our approximate value (i.e. 3.14 or $\frac{22}{7}$) of π in our formula. Conventionally, we choose to leave the answer in terms of π (we'll be following this convention here).

(ii) This time, we replace our radius with 5 m and use the same formula. This gives:

$$Circumference = 2\pi R = 2\pi \times 5\ m = 10\pi\ m$$

(iii) Again, we replace our radius with 0.5 cm, giving:

$$Circumference = 2\pi R = 2\pi \times 0.5\ cm = \pi\ cm$$

(b) (i) Here, we're given the **diameter** instead of the radius. This means that we're given $\mathbf{2}R$ instead of just the radius. This means that our formula would be:

$$Circumference = \pi \times (diameter) = \pi \times 1\ cm = \pi\ cm$$

(ii) We follow the same method as in part(a) and use 0.5 cm for our diameter, giving:

$$Circumference = \pi \times (diameter) = \pi \times 0.5\ cm = \frac{\pi}{2}\ cm = 0.5\pi\ cm$$

(iii) Again, we follow the same method as in part(a). Notice that our units this time are in **meters** rather than in centimeters. Plugging in our diameter into the circumference formula gives:

$$Circumference = \pi \times (diameter) = \pi \times 10\ m = 10\pi\ m$$

(iv) Interestingly, here we're given *any* possible diameter d in any possible units (which we write simply as *units*. We treat d just like we would any other number and plug it into our circumference formula:

$$Circumference = \pi \times (diameter) = \pi \times d\ units = \pi d\ units$$

Example 2

(a) Here, we're asked to find the equivalents of your standard 90°, 45°, 180° and 0° angles if a circle is defined to contain 40° instead of your usual 360°. The original angles and they're equivalents (when a circle contains 40°) are shown in the table below.

Table 1: Example 2(a)

Original Angle	Equivalent
90°	10°
45°	5°
180°	20°
0°	0°

For example, 90° when a circle contains 360° actually means $\frac{90}{360}$ths of a full turn i.e. a quarter turn. So, if a circle contains 40°, then a quarter turn must be $\frac{40^\circ}{4} = 10^\circ$.

Similarly, 45° when a circle contains 360° actually means $\frac{45}{360}$ths of a full turn and so on for each of the other angles.

An interesting thing to note is that 0° is the same regardless of whether a circle contains 360° or 40°. This makes sense because if you didn't turn at all, then measuring your angles differently will not affect whether you turned or not.

(b) We follow a similar method as in part(a) and write our results as shown in the table below. Remember that now we're asked to find the equivalent angles when our circle contains 100° instead of 360°.

Table 2: Example 2(b)

Original Angle	Equivalent
90°	25°
45°	12.5°
180°	50°
0°	0°

Notice that 0° remains unchanged as before.

(c) Now we're asked to find the equivalent angles if our circle contains 180° instead of 360°. We follow a similar method as in part(a) and write our results as shown in the table below.

Table 3: Example 2(c)

Original Angle	Equivalent
90°	45°
45°	22.5°
180°	90°
0°	0°

Again, we see that 0° remains unchanged.

Example 3

This example involves a very straightforward use of the arc length formula given on Page 5 of the handout. We're given that our circle has a radius of 4 units. This means that our circumference must be $2\pi \times 4 = 8\pi$ units.

(a) Here we're given that the fraction of the circle that Sebastian covers is $\frac{1}{8}$. Plugging this into our formula we get:

$$\textit{Arc Length} = \textit{(Circumference)} \times \textit{(Fraction of the circle that Sebastian covers)} =$$
$$8\pi \times \frac{1}{8} = \pi \textit{ units}$$

(b) Here our fraction is changed to $\frac{3}{4}$th of the entire circle. This means that the arc length is:

$$\textit{Arc Length} = \textit{(Circumference)} \times \textit{(Fraction of the circle that Sebastian covers)} = 8\pi \times \frac{3}{4} = 6\pi \ units$$

(c) Again, our fraction is simply changed to $\frac{2}{3}$rd of the entire circle. This means that the arc length is:

$$\textit{Arc Length} = \textit{(Circumference)} \times \textit{(Fraction of the circle that Sebastian covers)} = 8\pi \times \frac{2}{3} = \frac{16\pi}{3} \ units$$

Notice that we left our answer as a fraction (instead of decimal form), following our convention. Both the decimal and fraction forms are correct answers.

Example 4

In this example, we'll be using the arc length formula as on Page 6 in the handout. This formula helps us find the arc length if the angle subtended by the arc at the center of the circle is given (in degrees).

(a) We're given that the radius of the circle is 4 units, which means that the circumference must be:

$$2\pi R = 2\pi \times 4 = 8\pi \ units$$

Now we plug in our circumference and the angle subtended by the arc at the center (given to be 45°) into our arc length formula:

$$\textit{Arc Length} = \frac{\textit{Angle subtended by the arc at the center}}{360^\circ} \times Circumference = \frac{45^\circ}{360^\circ} \times 8\pi = \pi \ units$$

(b) We use the same formula as in part(a) except that this time, our radius is π units and the angle subtended by the arc at the center is 30°.

A radius of π units is just another radius. You treat it just like you would any other number. This means that our circumference is $2\pi R = 2\pi \times \pi = 2\pi^2 \ units$.

Plugging this and our angle into the formula gives:

$$\textit{Arc Length} = \frac{\textit{Angle subtended by the arc at the center}}{360^\circ} \times Circumference = \frac{30^\circ}{360^\circ} \times 2\pi^2 = \frac{\pi^2}{6} \ units$$

Example 5

(a) In this example, we're asked to convert degrees to radians. The way we do this is by using the second formula given on Page 9 of the handout.

(i) We're asked to convert $1°$ into radians. To do this we use our formula:

$$\theta(radians) = \pi \frac{x(degrees)}{180°} = \pi \frac{1°}{180°} = \frac{\pi}{180}\ radians$$

If we wanted to, we could divide the $\frac{\pi}{180}$ to get approximately 0.017 radians.

(ii) We follow the same method as in part(a) to get:

$$\theta(radians) = \pi \frac{x(degrees)}{180°} = \pi \frac{360°}{180°} = 2\pi\ radians$$

Indirectly, this means that if we took a piece of string that's equal to the radius of **any** circle we wanted, it would fit *just over* 6 times around the edge of the circle (or exactly 2π times).

(iii) We use our formula once again:

$$\theta(radians) = \pi \frac{x(degrees)}{180°} = \pi \frac{90°}{180°} = \frac{\pi}{2}\ radians$$

(iv) $$\theta(radians) = \pi \frac{x(degrees)}{180°} = \pi \frac{180°}{180°} = \pi\ radians$$

(b) Now we're asked to go the opposite way and convert from radians back into degrees. We use the third formula on Page 9 to do this.

(i) To convert 1 radian into degrees we use our formula:

$$x(degrees) = \frac{180°}{\pi}\theta(radians) = \frac{180°}{\pi} \times 1 = \frac{180}{\pi}°$$

In decimal form, this is approximately $57.3°$.

(ii) We use the same formula again to get:

$$x(degrees) = \frac{180°}{\pi}\theta(radians) = \frac{180°}{\pi} \times \pi = 180°$$

(iii) Similarly:

$$x(degrees) = \frac{180°}{\pi}\theta(radians) = \frac{180°}{\pi} \times 2\pi = 360°$$

(iv) Finally:

$$x(degrees) = \frac{180°}{\pi}\theta(radians) = \frac{180°}{\pi} \times \frac{\pi}{3} = 60°$$

Problem Set

1. In each case, we'll be using our standard circumference formula:

$$Circumference = 2\pi R = \pi D$$

   (a) We're given that the radius is 3 units. This means that:

$$Circumference = 2\pi R = 2\pi \times 3 = 6\pi \ units$$

   (b) Similarly, here we've got a radius of 2.5 units.

$$Circumference = 2\pi R = 2\pi \times 2.5 = 5\pi \ units$$

   (c) Our **diameter** here is 4 units.

$$Circumference = \pi D = \pi \times 4 = 4\pi \ units$$

   (d) This one's different in that our diameter is 2π units. Remember, we treat π just like any other number so our method is no different.

$$Circumference = \pi D = \pi \times 2\pi = 2\pi^2 \ units$$

2. (a) We're given that the radius of the circle is 2 units, which means that the circumference must be $2\pi \times 2 = 4\pi$ units.

   When the angle subtended by the arc at the center is given in degrees, we use the following arc length formula:

$$Arc\ Length = \frac{Angle\ subtended\ by\ the\ arc\ at\ the\ center}{360^\circ} \times Circumference$$

   Here our angle is 60°. This means that the arc length must be:

$$Arc\ Length = \frac{60^\circ}{360^\circ} \times 4\pi = \frac{2\pi}{3} \ units$$

   (b) Now we're given the angle in radians instead of degrees. As mentioned in class, one of the key benefits of using radians is that our equations and formulas become simpler. In this case, we'll use the following arc length formula (as on Page 8 in the handout):

$$Arc\ Length = R\theta$$

Given that our **diameter** is 2 units, our radius must be half of it i.e. $R = \frac{2}{2} = 1$ unit. This means that our arc length is:

$$Arc\ Length = R\theta = 1 \times 1 = 1\ unit$$

So if we've got an arc that's exactly the same length as the radius of the circle that it's a part of, then the angle the arc subtends at the center is 1 radian.

3. (a) A square of side 4 units has a perimeter of $4 \times 4 = 16$ units.

   (b) A circle of radius 4 units has a perimeter (or circumference) of $2\pi \times R = 2\pi \times 4 = 8\pi$ units.

   (c) An equilateral triangle of side 4 units has a perimeter of $3 \times 4 = 12$ units.

   8π units is approximately 25.12 units. This means that the circle has the greatest perimeter.

4. **Note:** Since the value of π you're using is only approximate, don't worry if your answers are off by a slight amount. Additionally, calculators could also give you slightly different answers.

   (a) The area of a circle of radius R is given by the formula:

   $$Area = \pi R^2$$

   If we're given the perimeter (or circumference), we can use our circumference formula *backwards* to find the radius of the circle.

   First, we write the circumference formula:

   $$Circumference = 2\pi R$$

   Now we're given that the circumference is 12 units. This means that:

   $$12\ units = 2\pi R$$

   So if we divide 12 units by 2π, we should get our radius. In other words:

   $$\frac{12}{2\pi} = R$$

   $$\frac{6}{\pi}\ units = R$$

   $\frac{6}{\pi}$ is approximately 1.91 units. This means that our radius is 1.91 units. Now we can find the area of the circle pretty easily:

   $$Area = \pi R^2 = \pi \times 1.91 \times 1.91 = 11.45\ sq.units$$

(b) If a square has a perimeter of 12 units, it means that:

$$12\ units = 4 \times (side)$$

This means that if we divide 12 units by 4, we'll get the side length of our square:

$$\frac{12}{4} = side$$

$$3\ units = side$$

A square of side 3 units has an area of:

$$Area = (side)^2 = 3^2 = 3 \times 3 = 9\ sq.units$$

(c) We can use the formula for the area of a triangle to (given in the hint) to find the area of our equilateral triangle. We know that the base of our equilateral triangle is $\frac{12}{3} = 4$ units.

To find the height, we need to use the Pythagorean theorem (as shown in the Figure).

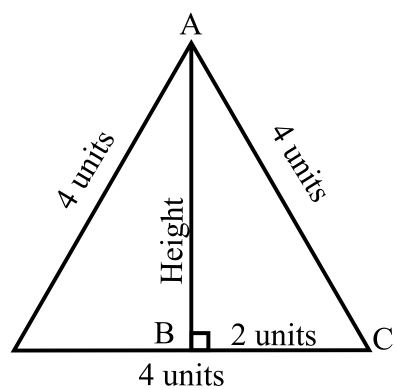

Figure 1: Using the Pythagorean theorem to find the height of the triangle

The height is therefore $\sqrt{4^2 - 2^2} = \sqrt{12}$ units. $\sqrt{12}$ units is approximately 3.46 units. So the height of our equilateral triangle is 3.46 units.

We can plug the height and the base into our formula to get our area:

$$Area = \frac{1}{2} \times (base) \times (height) = \frac{1}{2} \times 4 \times 3.46 = 6.92\ sq.units$$

It's clear that the circle has the largest area amongst the three at 11.45 sq. units.

In fact, this can be proved to be general i.e. the figure enclosing the greatest area for a given perimeter, is the circle. This is the solution to the **isoperimetric problem** ('iso' - same).

There's an interesting story about an Egyptian Queen who was allowed to own as much land as she could enclose using a buffalo hide. She quickly found out that cutting the hide into thin strips and joining them end to end to form a circle was her best bet to get as much land as she possibly could.

5. We're asked to find the number of **complete** rotations that Jenson's wheel makes before it comes to a stop.

   First, we know that the wheel comes to a stop after 100 feet. One foot contains 12" (inches) which means that the wheel rolls for a distance of $100 \times 12 = 1200$" (inches).

   We also know that the **diameter** of the wheel is 20" (inches). The distance a wheel covers when it makes a complete rotation is equal to the circumference of the wheel. This is because the wheel rolls exactly along it's edge.

   The circumference of Jenson's wheel must then be $\pi D = \pi \times 20 = 20\pi$"(inches).

   So to find the number of rotations the wheel makes before it stops (i.e. when the wheel travels 1200") we divide the distance the wheel travels before it stops, by the distance traveled by the wheel in one rotation. In other words:

   $$\#\ of\ rotations = \frac{Distance\ covered\ by\ wheel\ before\ stopping}{Distance\ traveled\ by\ wheel\ in\ one\ rotation}$$

   As mentioned earlier, the distance traveled by the wheel in one rotation is simply it's circumference. This gives:

   $$\#\ of\ rotations = \frac{Distance\ covered\ by\ wheel\ before\ stopping}{Circumference}$$

   Plugging in our values for the distance traveled by the wheel and the circumference, we get:

   $$\#\ of\ rotations = \frac{1200"}{20\pi"} = \frac{60}{\pi}\ rotations$$

   This is approximately 19.10 rotations. Since the question asks for the number of **complete** rotations, we say that the wheel completes **19** rotations before coming to a stop.

6. This problem involves a rather straightforward use of the formula for the area of a circle.

   First, we're given that Farmer John has a circular field of diameter 30 m with a smaller circular pen of diameter 10 m *within* it.

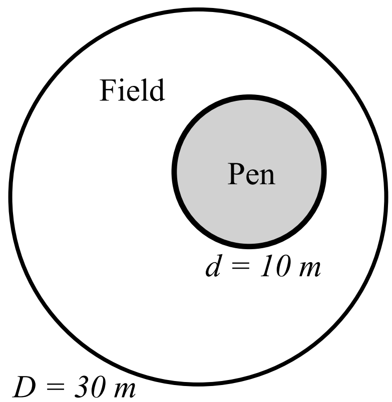

Figure 2: A diagram showing Farmer John's field and the pen

   We're asked to find the total grazing area available to the cows i.e. the area in between the two circles.

   From the figure, it's clear that the area in between the circles is the difference between the area of the field and the pen i.e.:

$$Grazing\ Area = Area\ of\ Field - Area\ of\ Pen$$

   The area of the field is simply π times the radius of the field squared. We know that the diameter of the field is 30 m, which means that the radius of the field is $R = \frac{30}{2} = 15\ m$. This means that the area of the field is:

$$Area\ of\ Field = \pi R^2 = \pi \times 15^2 = 225\pi\ m^2$$

   Similarly, the diameter of the pen is 10 m which means that it's radius is $r = \frac{10}{2} = 5\ m$. This means that the area of the pen is:

$$Area\ of\ Pen = \pi r^2 = \pi \times 5^2 = 25\pi\ m^2$$

Therefore the grazing area must be:

$$Grazing\ Area = Area\ of\ Field - Area\ of\ Pen = 225\pi - 25\pi = 200\pi\ m^2$$

In decimals, this is approximately 628 m^2.

So the total grazing available to the cows is approximately **628** m^2.