UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

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Probability

Introduction

Probability is a measure of how *likely* something is to occur. For example, you might have heard the weather forecast say that there's a 90% chance of thunderstorms tomorrow. Probability quantifies (or puts into numbers) the likelihood of some event happening. So although statements like "There's a good chance I won't go to that party" talk about chance or likelihood, they do not talk about probability in a mathematical sense.

The branch of mathematics that deals with probability is called *Probability Theory*.

This week, we'll take a look at some important rules we follow when we try and calculate probabilities and a few interesting applications of probability theory.

Definitions of Probability

Historically, the probability of some event A was defined as:

$$\textit{Probability of event A happening} = P(A) = \frac{\textit{\# of possible ways in which A can occur}}{\textit{Total \# of possible outcomes}}$$

This definition is called the **Classical Definition** of probability.

As a simple example, consider a coin which is *unbiased* (i.e. it has an equal chance of landing heads or tails). There are 2 possible outcomes H (heads) or T (tails). Let's say we want to find the probability of landing heads ($P(H)$). There's only 1 way in which we can get a tail. So we say:

$$\textit{Probability of getting a tail} = P(T) = \frac{1}{2}$$

Similarly, there's only 1 way in which we can land a head. So we say:

$$\textit{Probability of getting a head} = P(H) = \frac{1}{2}$$

Example 1:

(a) What's the probability of getting a 4 when rolling a standard *unbiased* 6-sided die?

(b) A bag contains 32 balls - 16 red, 10 black and the remaining purple. If you pick one from the bag without looking (i.e. *at random*), what's the probability of picking a purple ball?

The classical definition of probability arose from the study of gambling in the 17^{th} century. In fact, two people - Blaise Pascal and Pierre de Fermat - were responsible for much of the early advances in classical probability theory.

There's another slightly different definition of probability. This definition states that the probability of an event A happening, is:

$$\textit{Probability of event A happening} = P(A) = \frac{\textit{\# of trials in which A occurs}}{\textit{Total \# of trials performed}}$$

This definition is often called the **Frequentist** or **Experimental Definition** of probability.

This definition is equivalent to our earlier classical definition. The key difference is in how we interpret the probability.

Going back to our coin flipping example, according to our classical definition, a probability of $\frac{1}{2}$ for getting a tail means that there's 1 way in which we can land a tail out of 2 possible ways.

On the other hand, according to our frequentist definition, a probability of $\frac{1}{2}$ for getting a tail means that if we flip a coin a large number of times, we expect to land tails half the time.

See how although we get the same answer with both definitions, the interpretation of our answer depends on which definition we use.

Another key point about the experimental definition of probability, is that certain events cannot be assigned probabilities. For example, if you are a frequentist (i.e. follow the experimental definition of probability), the question “What is the probability that there is life on Mars?” wouldn’t make any sense to you. This is because according to your definition of probability, you need to check for life on Mars multiple times for you to calculate this probability. You can’t really do this because this would require you to go to all possible universes and check for life on Mars.

The key point here is that the experimental definition of probability works only with *repeatable* events. So questions such as “What’s the probability that there’s a ghost in the attic?” cannot be meaningfully answered because they’re not repeatable events in the usual sense.

For the purposes of this week’s examples and problem set, we can use either of our two definitions of probability.

Sample Space

Notice how our definitions of probability depend on the total number of trials or possible outcomes. The set of all possible outcomes of any experiment we do is called the **Sample Space** of that experiment.

For example, the experiment of flipping a coin has 2 possible outcomes H or T. We write the set of all possible outcomes or *Sample Space* as:

$$S = \{H, T\}$$

Similarly, if you were to flip a coin two times in a row, there would be 4 possible outcomes. The sample space for this experiment would be:

$$S = \{HH, HT, TH, TT\}$$

Example 2: Write the sample space for the following experiments:

(a) Flipping two *unbiased* coins at the same time.

(b) Flipping an *unbiased* coin 3 times.

(c) Rolling a standard, *fair* 6-sided die *twice*.

Why do we bother writing down this sample space? Well, because the number of elements in the sample space is the total number of possible outcomes. We need the total number of possible outcomes to calculate the probability of events we're interested in.

Of course it's important to realize that we don't bother writing down the sample space when it's too large. For example, think of rolling a 6-sided die 10 times. There are 6^{10} possible outcomes and it's not realistic to list each and every one of them. We deal with these sample spaces by using counting techniques (i.e. Permutations & Combinations, we won't be covering these).

Independent & Mutually Exclusive Events

Say you're flipping an unbiased coin 3 times as in Example 2 (b). Notice that the probability of getting a tail on the first flip is $\frac{1}{2}$ as we found before. The probability of getting a tail on the second flip is also $\frac{1}{2}$. This is because the first flip has no influence (as far as we can tell) on the second flip.

This means that if a flip a coin n times, each individual flip has a $\frac{1}{2}$ probability of landing a tail (or alternatively, a head). In other words, the probability of landing a tail on any particular flip is **independent** of how many flips you did and how many you're going to do.

So if you flip a coin twice, each flip is called an **independent event** i.e. the probability of landing a tail on the second flip is independent of whatever you got on the first.

Also, it's obvious that you can't get both a head and a tail at the same time if you flip the coin only once. In other words, the events of getting a tail or a head are **mutually exclusive** i.e. they cannot occur at the same time during the same experiment.

Two events can be mutually exclusive and independent at the same time.

Example 3:

(a) Which of the following pairs of events are mutually exclusive?

   (i) Picking a King and an Ace from a deck of 52 cards.

   (ii) Picking a red card and a diamond card from a deck of 52 cards.

   (iii) Getting at least one head and at least one tail after 3 consecutive flips of a coin.

(b) Which of the following pairs of events are independent?

   (i) Flipping a coin and then rolling a 6-sided die.

   (ii) Picking two red balls consecutively from a bag containing 10 red and 12 black balls (with replacement).

   (iii) Picking an ace followed by a king from a deck of 52 cards (with replacement).

Addition & Multiplication Rules

Now that we know how to calculate basic probabilities, there are two rules that help us calculate probabilities of more complicated and interesting events.

Addition Rule

The addition rule for probability tells you that if you want to find the probability one or the other mutually exclusive event occurring, you simply add the probabilities of each event.

For example, if you want to find the probability that you get a tail or a head when flipping a coin once, you write:

$$\textit{Probability of getting a T or an H} = P(H\,or\,T) = P(H) + P(T) = \tfrac{1}{2} + \tfrac{1}{2} = \mathbf{1}$$

This makes sense, because if we use our classical definition of probability, we see that there are two ways in which we can get a T or an H when we flip a coin once. Also, there are only two possible outcomes. This means that:

$$\textit{Probability of getting a T or an H} = P(H\,or\,T) = \frac{\textit{\# of ways to get H or T}}{\textit{Total \# of possible outcomes}} = \tfrac{2}{2} = \mathbf{1}$$

Notice that a probability of 1 means that the event in question is **certain** to happen, whereas a probability of zero means that it's **impossible** for the event to happen.

Also, **the sum of probabilities of all possible outcomes of an experiment should be 1**. This is because you're *certain* that *any one* of the possible outcomes should occur - you can't flip a coin and expect an outcome other than an H or a T.

Similarly, say there's 3 possible weather conditions - rainy, sunny and snowy - each with a 1 in 3 chance of happening on any given day. How would you find the probability that it's either rainy or snowy on a particular day (assuming that any particular day can have only one weather condition)?

First, we notice that our assumption means that the weather conditions are mutually exclusive i.e. you can't have two weather conditions on the same day. This means that we can use our addition rule to calculate the probability:

$$\textit{Probability that it is either rainy or snowy on a given day} = P(\textit{Rainy or Snowy}) = P(\textit{Rainy}) + P(\textit{Snowy}) = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}$$

Multiplication Rule

The second rule applies to events that occur one after the other. The multiplication rule for probability says that if you want to find the probability of two independent events occurring one after the other, you simply multiply the probabilities of each event.

For example, if you wanted to find the probability that you get two tails in a row when flipping a coin twice, you write:

$$\textit{Probability of getting two Ts in a row} = P(T) \times P(T) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$

Similarly, if you wanted to find the probability that you get two sixes when rolling a pair of dice, you'd write:

$$\textit{Probability of getting to 6s} = P(\textit{getting a 6}) \times P(\textit{getting a 6}) = \tfrac{1}{6} \times \tfrac{1}{6} = \tfrac{1}{36}$$

Example 4:

What is the probability of picking an ace followed by a king from a deck of 52 cards (without replacement)? What is the probability of picking an ace followed by a king from the same deck if the first card picked is put back into the deck before picking the second (i.e. with replacement)?

The addition and multiplication rules come from similar **fundamental counting principles**.

To see how the addition rule came about, say you want to get to Toronto. There are 3 busses and one train from your city to Toronto. If you wanted to find the total number of ways you can get to Toronto, you'd simply add the 3 ways (due to the busses) and the 1 way (due to the train) to get **4** ways in total. Notice that you can either take a bus or a train but not both at the same time.

Similarly, say you have 3 shirts and 6 pairs of jeans. To find how many combinations you can make in total, you'd need to use the multiplication rule. For each shirt, you can wear any one of your 6 pairs of jeans. So if you wear the first shirt, you have 6 possible combinations (1 for each pair of jeans).

Similarly, you have 6 possible combinations each for the second and third shirts. This gives you a total of $6 \times 3 =$ **18** possible combinations.

The addition and multiplication rules are fundamental consequences of the **modern definition of probability** which uses the **Kolmogorov Axioms**. They are 3 fundamental rules from which all the properties of probability can be deduced.

Probability Theory and **Statistics** (another branch of mathematics that involves experiments and data collection) are important because they're often the most misused and misunderstood.

As an example of this, consider a running race between you and your friend (who goes to another school). You win the race and announce: "I came first in an inter-school running race", to your other friends at school.

Your friend on the other hand, goes back to his/her school and announces: "I came second in an inter-school running race". See the problem here? There were only two people in the race! So although your friend was being truthful, he/she didn't give any context for their announcement. This means that someone with no background information would think that your friend did well on the race, even though he/she actually lost.

Problem Set

1. What is the probability of getting a number between 1 and 6 (including both 1 and 6) when rolling a 6-sided die?

2. What is the probability of getting a 9 when rolling a 6-sided die?

3. You do two experiments in a row: First you flip a coin and then you roll a 7-sided die.

   (a) Write the sample space for each experiment individually.

   (b) Write the sample space for the experiment of flipping a coin then rolling a 7-sided die.

   Hint: This is different from part(a). This question asks you to write all possible outcomes of flipping a coin then rolling a 7-sided die.

   (c) What is the probability of getting a T and then rolling a 4?

4. You're having buttered toast for breakfast. Unfortunately, you slip and fall while carrying your plate of toast to the living room. Since the toast is unbalanced i.e. heavier on the buttered side, there's a $\frac{1}{8}$ chance that it will land butter side up. What is the probability that your toast lands butter side down? *(Assume your toast can land only butter side up or down - not on it's side etc.)*

   Bonus: If you had two pieces of buttered toast (instead of just one), what's the probability that both of them land butter side up?

5. What's the probability of hitting the bullseye on a dart board if the bullseye has a radius of 1 cm and the board, a radius of 18 cm?

   Hint: Area of a circle $= \pi R^2$

6. You're on a game show where you're asked to pick one of three closed doors. Behind two of the three doors, there are goats. But behind one of them, there's a brand new car.

   (a) What is the probability of winning the car?

   You've now picked a door. The gameshow host opens one of the doors you didn't pick and reveals a goat. Now there are two closed doors and one open door with a goat. The host gives you one last chance to change your door.

   (b) Should you change your mind and pick the other door? Why or why not?

   Hint: Does your probability of winning the car change when the host opens one of the doors? The answer to this question is the key to this problem.