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Series & Polygonal Numbers

Solutions

Example 1

(a) When you roll a standard *unbiased* 6-sided die, there are 6 possible outcomes. They are:

$$S = \{1, 2, 3, 4, 5, 6\}$$

In other words, S is the **sample space** for the experiment of rolling a 6-sided die once.

There's one outcome that gives you a 4 out of a total of 6 possible outcomes. This means that the probability of getting a 4 is:

$$P(\textit{getting a 4}) = P(4) = \frac{\textit{(\# of ways in which you can get a 4)}}{\textit{(Total \# of possible outcomes)}} = \frac{1}{6}$$

(b) We want to find the probability of picking a purple ball. To do this we need the total number of balls in the bag and the number of ways in which you can pick a purple ball.

The first one's easy - it's given that there are 32 balls in the bag.

To find the number of ways in which you can pick a purple ball, you just need to find the number of purple balls. There are 32 balls in total, with 16 red and 10 black. This means that the number of purple balls is:

$$\textit{(\# of purple balls)} = 32 - 16 - 10 = 6$$

This means that the probability of picking a purple ball *(at random)* from the bag is:

$$\textit{P(picking a purple ball)} = \frac{\textit{(\# of ways of picking a purple ball)}}{(Total\#ofpossibleoutcomes)} = \frac{6}{32} = \frac{3}{16}$$

It's not necessary to simplify the answer unless otherwise specified.

Example 2:

(a) The question asks you to write the sample space for the experiment of flipping two unbiased coins at the same time. Remember that the **sample space** is simply **the set of all possible outcomes**.

When flipping the first coin, there are two possible outcomes: H or T. Suppose you get H. Then there are another two possible outcomes when flipping the second coin: HH or HT i.e. you could get H first and H again, or H first and T second.

Similarly, if you'd got a T for your first flip, the possible outcomes for the second flip are: TH or TT.

So, the set of all possible outcomes (i.e. the sample space) of this experiment is:

$$S = \{HH, HT, TH, TT\}$$

Interestingly, if you wrote out the sample space for flipping a *single* coin *twice*, you'd get the same sample space. This has to do with the fact that flipping two coins is an **independent event**. In other words, the outcome of flipping one coin doesn't affect the outcome of the other and vice-versa.

(b) If you flip an unbiased coin 3 times, you have a total of 8 possible outcomes. This is because each flip has 2 outcomes. So when you flip a coin 3 times, you get $2 \times 2 \times 2 = 8$ outcomes.

It might be helpful to think of this experiment in the following way: for each outcome of the first flip, there are two possible outcomes for the second flip and for each outcome of the second flip, there are another two possible outcomes for the third flip.

So say you get H on your first flip. Your second flip could give you an H or a T. Now say, you got a T. Your third flip could give you another H or a T.

If you were to write all these combinations out, you'd get the sample space:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Again, it's interesting to note that this is the same sample space you'd get if you flipped 3 different coins at the same time.

(c) When you roll a die once, there are 6 possible outcomes. Rolling a die twice should then give you $6 \times 6 = 36$ possible outcomes.

This is because for each outcome of your first roll, you have a further 6 possible outcomes. You have 6 possible outcomes for your first roll and a further 6 possible outcomes for each of those, giving you a total of 36 possible outcomes.

You then write out the sample space as:

$S = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$

$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$

$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

Similar to our previous examples, this is the exact same sample space you'd get if you rolled two dice at the same time.

It's important to understand that although we write the outcomes as an ordered pair (i.e. (1, 3) etc.), they are **not** points in 2-D.

Example 3:

(a) Remember, mutually exclusive events are those which can't occur at the same time in the same experiment.

(i) These events are mutually exclusive. You can pick either a King or an Ace but not both at the same time.

(ii) These events are **not** mutually exclusive. You could pick a card which is both red and diamonds at the same time.

(iii) These events are also **not** mutually exclusive. To see why, look at Example 2 (c). You can have an outcome which has both at least 1 tail and at least 1 head at the same time.

(b) (i) This pair of events is independent. This is because the outcome of our coin flip doesn't affect the outcome of rolling the 6-sided die in any way (assuming that the coin and the die have not been tampered with).

(ii) This pair of events is also independent. This is because what ball you pick on your first try doesn't affect what ball you pick on your second. The outcome of one event does not affect the outcome of the other.

However, if we **did not** put back the first ball we picked, the total number of balls available changes. This affects the outcome when we pick our second ball. In such a case, the events would be **dependent**.

(iii) This pair of events is independent. This is because as long as you put back the first card you pick, you won't affect the outcome of your second pick, just like in part (ii).

If you didn't return the card (i.e. without replacement), you'd be affecting the outcome of your second pick which would mean that the events would be **dependent**.

Example 4:

The first part of the question asks you to find the probability of picking an Ace followed by a King from a deck of 52 cards (without replacement).

Notice that the fact that we don't put back our first card (i.e without replacement) means that the probability of picking a King (our second event), is affected by our first event. This means that they are **dependent** events.

However, we know that the probability of one event occurring after an other is simply the product of their probabilities (i.e. the **Multiplication Rule**).

So to find $P(picking\ an\ Ace\ followed\ by\ a\ King)$, we need to find $P(picking\ an\ Ace)$ and $P(picking\ a\ King)$.

The event of picking an ace occurs first so let's find the probability for that first.

There are 4 Aces in a deck of 52 cards. This means that the probability of picking an ace is:

$$P(picking\ an\ Ace) = \frac{(\#\ of\ ways\ of\ picking\ an\ Ace)}{(Total\ \#\ of\ possible\ outcomes)} = \frac{4}{52} = \frac{1}{13}$$

The event of picking a King occurs second. There are 4 Kings in a deck of 52 cards. But, we've only got 51 cards now - we didn't put our Ace back into the deck. This means that the probability of picking a King is:

$$P(\textit{picking a King}) = \frac{\textit{(\# of ways of picking a King)}}{\textit{(Total \# of possible outcomes)}} = \frac{4}{51}$$

Now that we have our two probabilities, we can find $P(\textit{picking an Ace followed by a King})$ by multiplying them according to our multiplication rule:

$$P(\textit{picking an Ace followed by a King}) = P(\textit{picking an Ace}) \times P(\textit{picking a King}) = \frac{1}{13} \times \frac{4}{51} = \frac{4}{663}$$

The second part of the question asks you to find the same probability if you put back the first card you picked before picking the second (i.e. with replacement).

In this case, $P(\textit{picking an Ace})$ doesn't change as you were picking an Ace first anyway. However, $P(\textit{picking a King})$ changes slightly.

Remember, there are 4 Kings in a deck of 52 cards. If you put back the first card that you picked, then you aren't changing the total number of cards available when you pick your second. This means that the probability of picking a King is also $\frac{4}{52}$.

Now, the events of picking an Ace and picking a King are **independent** events - the outcome of one doesn't affect the other.

We can still use the multiplication rule to find the probability we need:

$$P(\textit{picking an Ace followed by a King}) = P(\textit{picking an Ace}) \times P(\textit{picking a King}) = \frac{4}{52}\frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$$

Problem Set

1. $P(\textit{getting a number between 1 and 6}) = 1$

   This is because there are only 6 possible outcomes when rolling a 6-sided die. It is **certain** that you'll get a number between 1 and 6 (including both 1 and 6) if you roll a die. A probability of 1 indicates that the event is certain.

2. $P(\textit{getting a 9}) = 0$

   This is because you cannot get a 9 when rolling a die which has just 6 sides. It's **impossible**. We assign a probability of 0 to impossible events.

3. (a) Let's call the set S the sample space for the coin flip.

   $$S = \{H, T\}$$

   Let's call the set D the sample space for the die roll. Remember that our die is 7-sided (instead of the usual 6).

   $$D = \{1, 2, 3, 4, 5, 6, 7\}$$

   (b) Let's call the set A the sample space for the experiment of flipping a coin then rolling a 7-sided die.

   $A = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (H, 7),$
   $(T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6), (T, 7)\}$

   You can tell without counting the number of elements of A that there must be a total of 14 possible outcomes - 7 when you get H and another 7 when you get T.

   (c) We need to find $P(\textit{getting a T and then a 4})$. We can use our multiplication rule to find the answer.

   Getting a T on our coin flip and getting a 4 on our die roll are **independent** events. However, they are not **mutually exclusive** as they can occur at the same time in the same experiment.

   Since our first event is getting a T, we need $P(\textit{getting a T})$ or $P(T)$. The probability of getting a T must be $\frac{1}{2}$ as we've seen before.

   Now, we need $P(\textit{getting a 4})$. Since we're rolling a 7-sided die, there are 7 possible outcomes. Getting a 4 is just one possible outcome out of a possible 7. This means that:

   $$P(\textit{getting a 4}) = \frac{\textit{(\# of ways of getting a 4)}}{\textit{(Total \# of possible outcomes)}} = \frac{1}{7}$$

   Now to find $P(\textit{getting a T and then a 4})$, we can use the multiplication rule:

   $$P(\textit{getting a T and then getting a 4}) = P(T) \times P(\textit{getting a 4}) = \frac{1}{2} \times \frac{1}{7} = \frac{1}{14}$$

   You can verify your answer by looking at the sample space for this experiment (which we wrote out in part (b)). There's only 1 way of getting an outcome of $(T, 4)$.

4. The buttered toast in this question is like having a *biased* coin - it's more likely to fall on one side than the other. You're given that the probability of your toast landing butter side up is $\frac{1}{8}$ and asked to find the probability that it lands butter side down.

   You know that there's only two possible ways in which your toast can land - butter side up or butter side down. In other words, you're **certain** that your toast lands either butter side up or butter side down. This means that the probability of your toast landing butter side up **or** butter side down must be **1**.

   If we put this into math, we get:

$$P(\textit{toast lands butter side up } \boldsymbol{or} \textit{ butter side down}) = 1$$

   A piece of toast cannot land butter side up **and** butter side down at the same time i.e. they are **mutually exclusive** events. This means that:

$$P(\textit{toast lands butter side up } \boldsymbol{or} \textit{ butter side down}) = \\ P(\textit{butter side up}) + P(\textit{butter side down})$$

   This comes from the **addition rule** and is due to the fact that the events 'toast lands butter side up' and 'toast lands butter side down' are mutually exclusive and are the only possible outcomes of the experiment.

   If we plug in the probabilities we already know into the above equation, we should be able to find the probability that the toast lands butter side down:

$$1 = \tfrac{1}{8} + P(\textit{butter side down})$$

   This means that:

$$P(\textit{butter side down}) = 1 - \tfrac{1}{8} = \tfrac{7}{8}$$

   In other words, there's a very good chance that your toast lands butter side down.

   Bonus:

   Now you've got two pieces of toast and are asked to find the probability that they both land butter side up.

   You already know $P(\textit{butter side up})$ for a single piece of toast. Instead of writing out the sample space for this experiment, we can find $P(\textit{both butter side up})$ using the **multiplication rule**. This is because we're asked to find the probability of one event (the first piece landing butter side up) followed by another (the second piece landing butter side up).

The multiplication rule says that the probability of one event occurring after another is just the product of the probabilities of each of the events. In this case, the multiplication rule means that:

$$P(\textit{both butter side up}) = P(\textit{first one butter side up}) + P(\textit{second one butter side up})$$

$P(\textit{first one butter side up})$ is just $\frac{1}{8}$ (given in the question). We also know that $P(\textit{second one butter side up})$ must also be $\frac{1}{8}$ because the second piece is unaffected by how the first piece lands - they are **independent** events.

So now we can find the probability that both land butter side up:

$$P(\textit{both butter side up}) = \frac{1}{8} \times \frac{1}{8} = \frac{1}{64}$$

In other words, there's a very good chance that both your pieces of toast land butter side down. Unlucky.

5. This question is interesting because finding the probability of hitting the bullseye requires you to find the following fraction:

$$P(\textit{bullseye}) = \frac{(\# \textit{ of ways of hitting the bullseye})}{(\textit{Total } \# \textit{ of ways you can hit anywhere on the entire dartboard})}$$

The problem here is that there's countless ways in which you could hit the bullseye. There's also countless ways you can hit the entire dartboard. So how will we find the probability? It seems like we're stuck.

Interestingly, there's a neat trick that we can use to get ourselves out of this sticky situation. Notice that we're interested in finding the probability, which is **always a fraction between 0 and 1**. This means that we don't need to find the **exact** number of ways in which we can hit the bullseye or the number of ways we can hit the entire board - we just need to make sure that our fraction is correct.

Clearly everyone would agree that hitting the bullseye is much harder than hitting anywhere on the entire board? Why? Most people would say that the bullseye has a smaller area than the entire board.

This is where things get interesting. The # of ways of hitting the bullseye is **directly proportional** to the area of the bullseye. In other words, there are more ways of hitting the bullseye if we increase it's area.

A similar argument applies for the entire dartboard. The # of ways you can hit anywhere on the entire dartboard is also directly proportional to the area of the dartboard. In other words, there are more places for you to hit if you have a bigger board.

This means that we can use the areas of the bullseye and the dartboard instead of counting the exact # of ways of hitting the bullseye and anywhere on the entire board respectively.

This means that:

$$P(bullseye) = \frac{\textit{(Area of bullseye)}}{\textit{(Area of entire dartboard)}}$$

The area of a circle of radius R is just πR^2. This means that the area of the bullseye (which has a radius of 1 cm) is $\pi \times 1^2 = \pi\ cm^2$.

Similarly, the area of the entire dartboard is $\pi \times 18^2 = 324\pi\ cm^2$.

This means that the probability of hitting the bullseye is:

$$P(bullseye) = \frac{\pi}{324\pi} = \frac{1}{324}$$

If we want, we can convert this to a percentage by multiplying it by 100. This gives approximately 0.3% which is a very tiny chance. The probability of hitting the bullseye on a real dartboard is slightly higher because the board is slightly smaller.

6. This is a famous problem called the **Monty Hall Problem**, after the gameshow host Monty Hall.

   You're given three doors, two of which have goats behind them and one which has a car behind it. You don't know what's behind each door specifically, but you do know that there are two goats and a car.

   (a) You want to find the probability of winning the car or $P(car)$. You know that there are 3 possible outcomes in this experiment of picking a door at random.

   Since you've got 2 goats and 1 car, there's only 1 way of picking the door with the car behind it. This means that:

   $$P(car) = \frac{\textit{(\# of ways of picking the car)}}{\textit{(Total \# of possible outcomes)}} = \frac{1}{3}$$

   So the probability of picking the car is $\frac{1}{3}$.

(b) You're now told that you've picked a door. The gameshow host opens one of the doors you didn't pick to reveal a goat. You now have two closed doors, one of which is the one you picked. You're given a chance to change the door you picked.

The question asks you whether or not you should change your choice.

To answer this question, we need to understand what happens to $P(car)$ when the host opens one of the doors with a goat behind it.

A good number of people would say that it doesn't matter if you change your choice or not since $P(car)$ increases to $\frac{1}{2}$ or 50%. According to them, this is because the number of possible outcomes is now reduced to 2 instead of 3.

Although this answer sounds right, it's actually **wrong**.

The reason has to do with the fact that you made the choice when all three doors were closed i.e. there were 3 possible outcomes.

When you first pick your door, $P(car) = \frac{1}{3}$. This means that the car has a $\frac{2}{3}$ probability of being behind one of the doors you **didn't** pick.

Now, when the host opens one of the doors you didn't pick and reveals a goat, then there's a $\frac{2}{3}$ probability that the car is behind the other closed door.

Since there's a 2 in 3 chance that the car is behind the remaining closed door (that you didn't pick), the answer is that **you should change your choice every single time**. To see how this solution works, it might be helpful to think of a game with 100 doors, 99 goats and 1 car. As before, you pick a door at random. There's a $\frac{1}{100}$ or 1% chance that the car is behind the door you picked.

If the host opens 98 of the remaining doors and reveals goats behind each and every one of them, then there's a $\frac{99}{100}$ chance that the car is behind the last closed door that you didn't pick. So if you're given a chance to change your choice, you should in order to increase your chances of winning.