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Grades 7 & 8, Math Circles

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Graph Theory

Introduction

Graph Theory is the branch of mathematics that studies **graphs** which are diagrams showing objects and how they're related to each other. The ideas of graph theory are often used in computer science, physics, sociology, linguistics and biology. This week, we're going to look at a few interesting problems and how we can solve them using some intuitive rules we have about graphs.

Origins

Euler is a name that's familiar to a lot of people, even outside mathematics. The primary reason for this was due to his remarkable ability to isolate the core of a problem. When Euler lived in Königsberg (Kaliningrad) in Russia in the 17th century, an interesting problem which involved finding a path through the city and it's bridges came to his attention. This is the now famous **Seven Bridges of Königsberg** problem.

As shown in Figure 1, the city of Königsberg was situated on the banks of the Pregel river with 7 bridges across it, connecting different parts of the city.

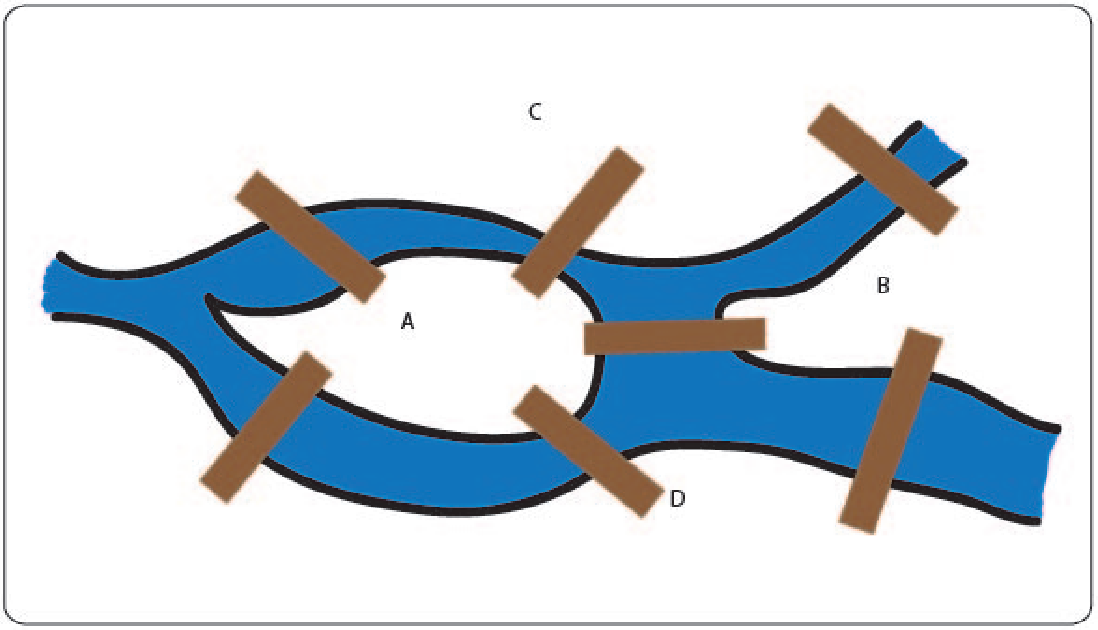


Figure 1: The Seven Bridges of Königsberg

The problem was this: "Is there a path you could take through the city that involved crossing each bridge **exactly once**?" The rules are that you're allowed to use only the bridges to cross the river and that you must cross a bridge if you're on one.

Euler is famous for having given the solution to this problem.

Simplifying The Problem

We want to try and solve the Königsberg bridge problem ourselves.

The first thing we could do, is to redraw the diagram so that things are clearer (Figure 2 - next page).

Each point (A, B, C etc.) represents one of the land masses in the problem.

Notice that our simpler diagram actually *loses* some information - the length of the bridges, the size of the land masses and even the straightness of the bridges themselves. So what's the point of drawing a simpler diagram if we lose so much information?

Well, it allows us to focus on the core of the problem (just like Euler). The core of this problem, is the land masses and the connections between them (i.e. the bridges and their start/end points). Any other information is irrelevant.

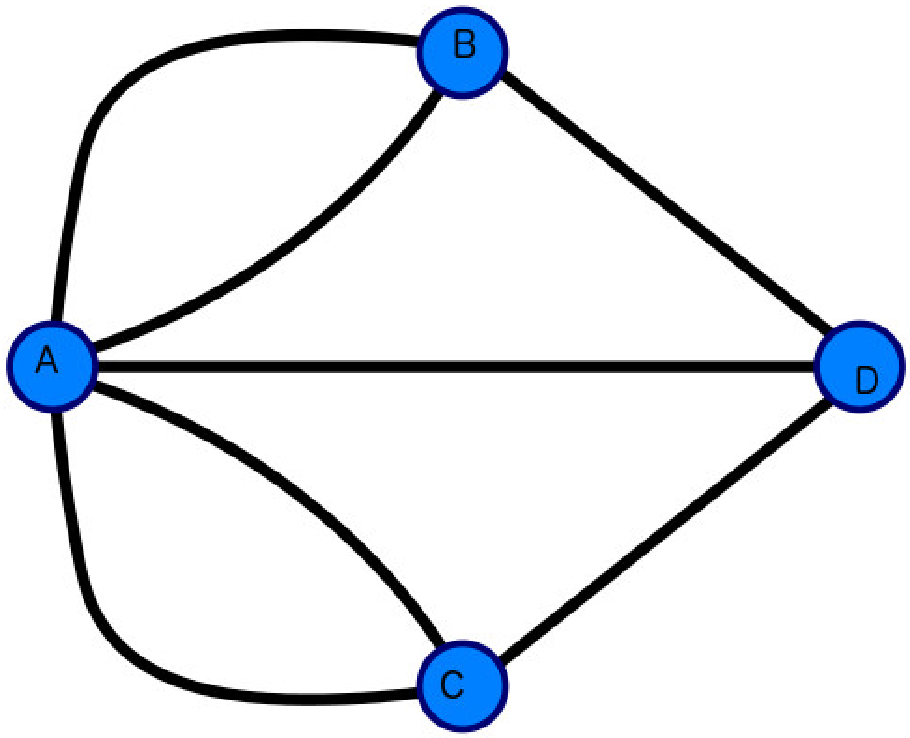


Figure 2: A graph representing our problem

To see this, imagine if one of the bridges was slightly longer than shown Figure 1. It wouldn't make a difference to our problem of walking through the city crossing each bridge exactly once. Similarly, imagine if one of the bridges was curved (without it's endpoints being changed). Once again, this makes no difference as far as our problem is concerned.

The simplified diagram we've just made is a **graph** representing our problem. Graphs only care about objects and the connections between them, just like how we don't care about the length or straightness of the bridges.

You could draw graphs any way you like - the key point is that the things you're interested in are connected correctly.

A Key Insight

We've simplified our problem by throwing away irrelevant information. What next?

Let's rephrase the problem in terms of our graph (Figure 2). The problem asks you if it's possible to start at any land mass (A, B, C or D) and end at any land mass (you're allowed to end at the one you started from), using all 7 bridges exactly once.

There are quite a lot of bridges in our problem. If we look at a simpler case with fewer bridges, we might be able to see something that could help us solve our problem.

Consider the case of 3 land masses and 2 bridges (Figure 3). Is it possible to start and end at any land masses using both bridges exactly once? Yes! It's easy - you start at X and cross the bridge to Y and then to Z (you could also go the other way).

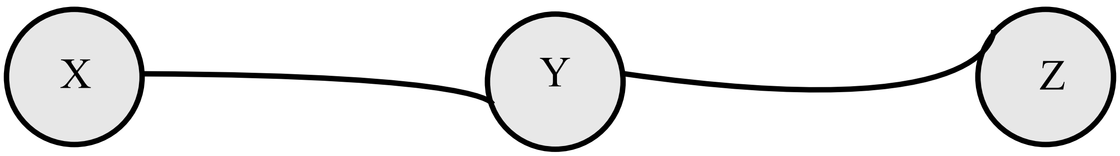


Figure 3: A simplified graph with 3 land masses and 2 bridges

Notice something interesting about Y - you enter it once and exit it once. In other words, the number of times you enter Y is equal to the number of times you exit Y. This isn't true for X or Z because they're either your starting or ending points. This means that **every land mass other than your starting and ending points must have an even number of bridges**.

Additionally, notice that your starting and ending points have an **odd** number of bridges (i.e. 1).

This is the key insight to solving our original problem with 7 bridges. Looking at Figure 2, all our land masses have an **odd** number of bridges. This means that all our land masses are either starting or ending points - we don't have any land masses that are neither (like Y from Figure 3).

This means that we can't find a path through the city that uses all 7 bridges exactly once.

Some Vocabulary

We've solved the 7 bridges problem. But the method in which we solved it - i.e. using a graph and noticing interesting properties about it - is general because it can be applied to other problems.

If we had 4 land masses and 6 bridges, we could use the same method to find a solution to a similar problem. Similarly, if we had 10 computers on a network with 13 connections between them, we could use graphs to answer interesting questions about the network.

So in order to talk about graphs in general, we need to learn some specialized vocabulary - we need other people to understand what we're talking about.

Vertices

When solving our problem, we drew dots/circles/points to represent the land masses. If we were solving a problem involving computers, the points would represent computers. In general, these dots/points/circles are called **vertices** (singular: **vertex**).

A vertex in a graph, represents one of the objects that we're interested in. Vertices are also often called **nodes**.

Edges

In our originial problem, land masses were connected to each other by bridges. We represented these as lines/curves between vertices on our graph. If we were solving a problem relating to a person's social network, each vertex would represent a friend/relative whereas each line/curve would indicate that two people know each other.

In general, these lines/curves are called **edges**. An edge in a graph represents a connection/relation between two vertices. We normally keep track of an edge in a graph by naming the vertices connected by the edge. So, edge $e = \{a, b\}$ refers to an edge that connects the vertices a and b.

In general, we talk about graphs in terms of their vertices and edges. So a graph $G = (V, E)$ refers to a graph with V vertices and E edges. Talking about graphs in this way leads to a few interesting things, one of which we'll see in the next section.

In our 7 bridges problem, we found that there was no path through the city that involved the use of each and every bridge exactly once. Our reasoning for this was that each vertex in our graph (Figure 2) had an odd number of edges. In graph theory speak, we mean that the **degree** of each vertex of our graph is odd. The degree of a vertex is just the number of edges connected to that vertex.

There are a few other important terms you will come across when learning graph theory. These are usually pretty abstract so we'll make notes of them as they pop up.

The Handshake Lemma

Example 1

Graph theory can tell you interesting things such as how many people shake hands at a party.

Sebastian, Lewis, Jenson, Susie and Nico are at a party. They've never met before so they all shake hands when they get there.

1. Draw a graph to represent this situation. What do the vertices and edges of this graph represent?
2. Find the degree of each vertex. How many times does each person shake hands?
3. What is the total number of handshakes that go around?

If you look closely at the above example, you'll notice something interesting. **The sum of the degrees of all the vertices is equal to twice the number of edges**.

This fact is called the **Handshake Lemma**.

How do we know this is true? Well, a single edge connects two vertices. So if we go through a graph and count all the edges, we should have twice as many *connections* between an edge and a vertex.

It's important to note that the handshake lemma applies to **undirected** graphs. An undirected graph is one where the edges are not *oriented*. This is in contrast to **directed** graphs where the edges have a definite orientation. We usually represent this fact by using a small arrowhead on the edge, indicating the orientation.

For an undirected graph, the edges $e(x, y)$ and $e(y, x)$ refer to the same thing.

Another thing to note from our example, is that every person shakes hands with everyone else. This is another way of saying that our graph representing the problem is **complete** i.e. every vertex is connected to every other vertex by a *unique* edge.

Planarity

Remember how we talk about graphs only in terms of their vertices and edges? An interesting consequence of this is that we're free to draw the graph in whichever way we like. This is of particular importance when talking about **planar graphs**.

Planar graphs are those which can be drawn without any of the edges crossing over each other. Interestingly enough, if a graph has edges crossing, it doesn't necessarily mean that the graph is **non-planar**. To see this, consider Example 2.

Example 2

Consider the two graphs shown below. Which one of them is planar?

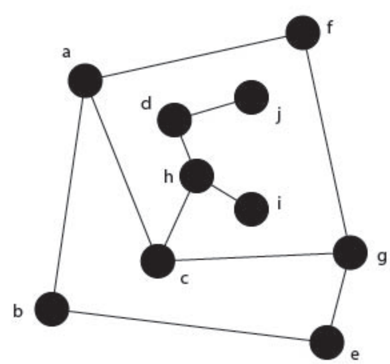


Figure 4: Example 2(a)

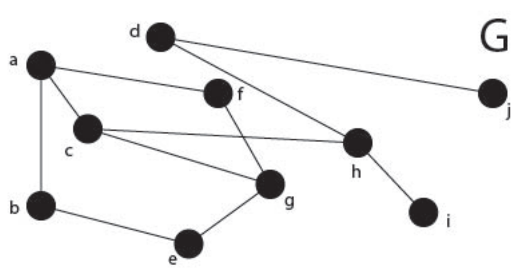


Figure 5: Example 2(b)

As seen in Example 2, checking for planarity can be tricky. Checking for non-planarity is even trickier - you could in theory keep rearranging the vertices forever! Thankfully, there's a helpful little theorem that tells us if a graph is non-planar or not.

Theorem 1 A graph with v vertices and e edges is non-planar if $e > (3v - 6)$.

Read the theorem again, it tells us that a graph is non-planar if the number of edges and vertices satisfy the above condition. It **DOES NOT** tell you if the graph is planar or not if the condition isn't satisfied. In other words, you could have a graph that is non-planar and doesn't satisfy the condition (your graph from Example 1 for instance).

So while Theorem 1 is helpful in certain situations, it is by no means comprehensive.

If a graph is planar, we can always draw it in a way that emphasizes this fact i.e. we can draw it as a **planar embedding**.

Example 3 According to Theorem 1, is the following graph planar or non-planar?

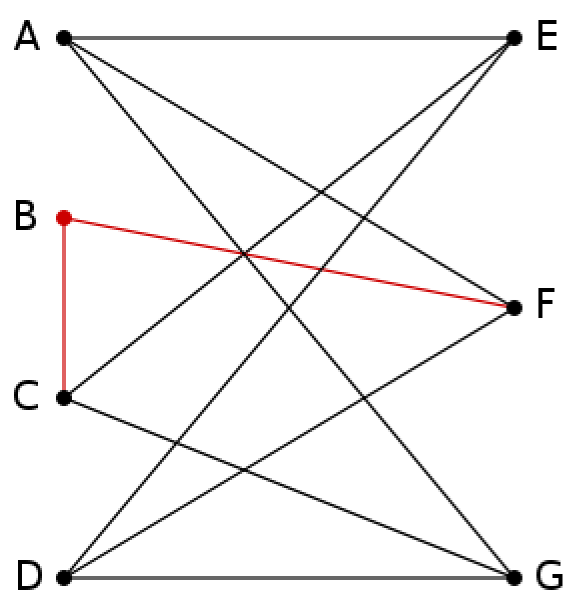


Figure 6: Example 3

Colouring

Example 4

Adjacent vertices of a graph are vertices that share a common edge. What is the fewest number of colour you need to colour each vertex of the following graph, such that **no two adjacent vertices have the same colour**?

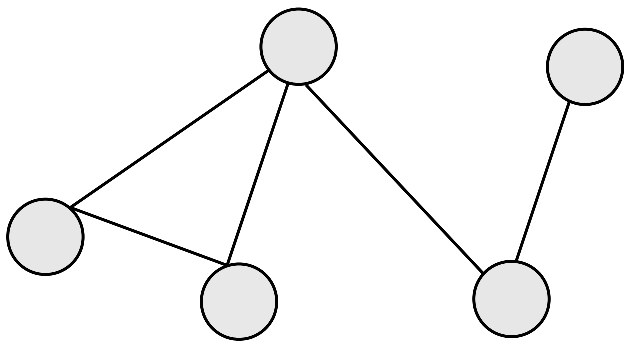


Figure 7: Example 4

Colouring is an important problem in graph theory. It refers to assigning a *colour* (or number) to each vertex such that no two adjacent vertices have the same colour. In fact, one of the biggest problems in math is the **Four Colour Theorem** which was only proven to be true fairly recently.

Notice that colouring doesn't really mean assigning a colour to each vertex. It could just as well be a number, or a letter.

Example 5

Talking about colouring a graph can lead to some interesting ideas. Consider the graph shown in Figure 7 below. What is the least number of colours needed to colour this graph (i.e. the **chromatic number**)?

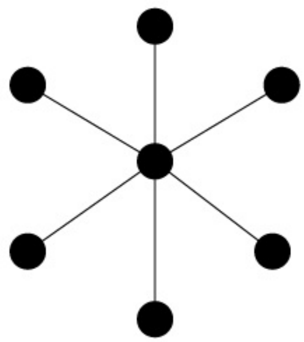


Figure 8: Example 5

The graph in Example 4 is an example of a **bipartite** graph. A bipartite graph is one whose vertices can be split into two groups (say A and B) such that every edge has endpoints at vertices in each group (see Figure 8).

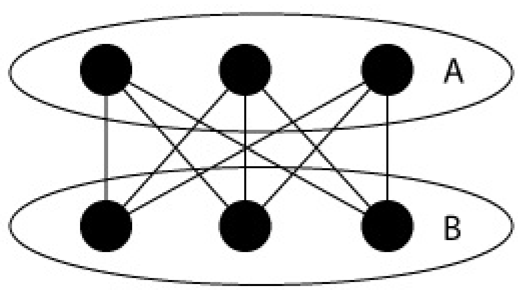


Figure 9: A bipartite graph

Bipartite graphs are interesting because they can be coloured completely by using just *two* colours. The reason for this is because there are no adjacent vertices in a group of vertices in a bipartite graph. This means that you only need two colours to colour the entire graph - one for each group.

Graph colouring has uses in a particular kind of problem called a **Scheduling Problem**. In the simplest case of such a problem, there's usually a number of tasks or jobs that need to be completed. Each one can be assigned a time slot.

Different jobs could *conflict* with each other in that they can't be done at the same time i.e. cannot be assigned the same time slot. Such a graph would have it's vertices representing the jobs in question and it's edges representing conflicting jobs.

The fewest number of colours required to colour such a graph i.e. the **chromatic number** of the graph, would in this case be the minimum number of time slots required to finish all jobs without any conflicts.

Colouring has applications in solving Sudoku puzzles as well!

Problem Set

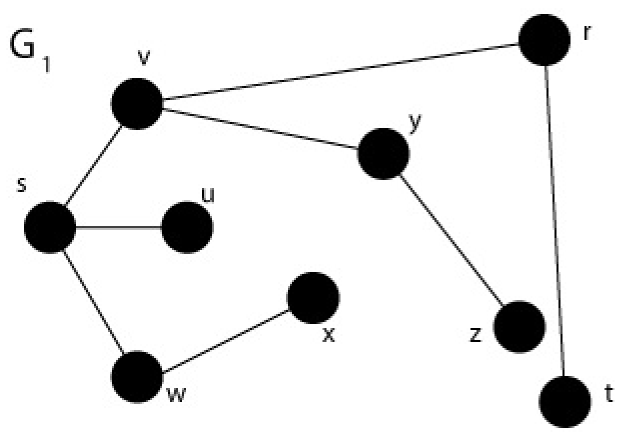
1. For each graph below, answer the following:

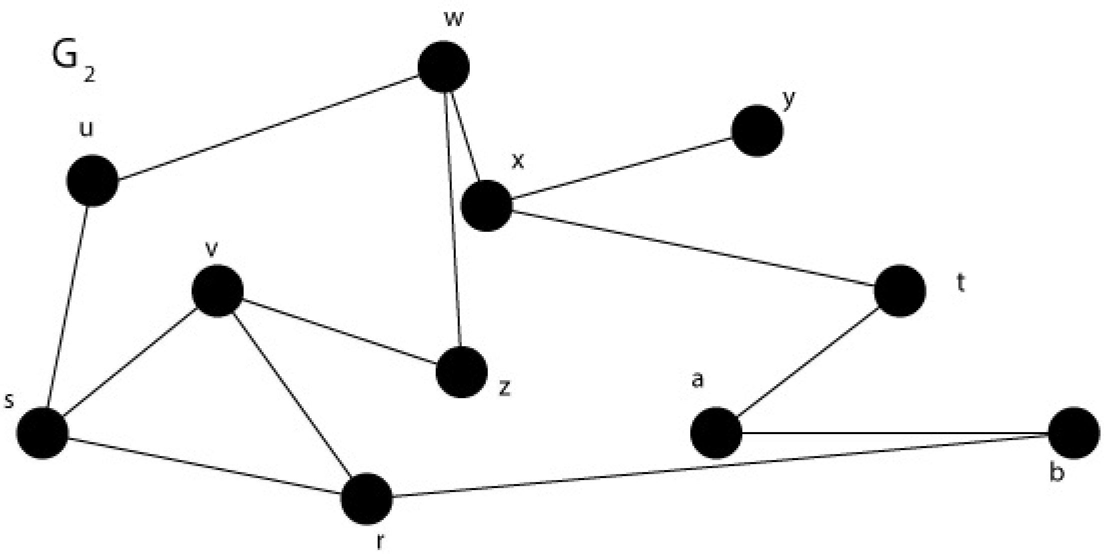
   (a) Write the number of edges and vertices. Write the graph in standard notation (i.e. in the form $G(V, E)$ where V is the number of vertices and E is the number of edges.).

   (b) List the vertices adjacent to (neighbors) of the vertex v.

   (c) Write the degree of each of the vertices $\{u, v, r\}$.

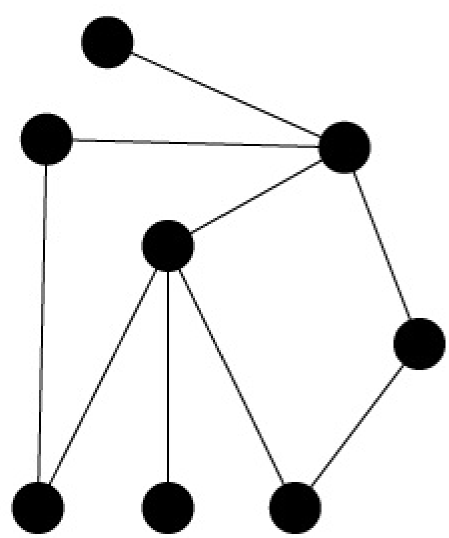
   (d) Are these graphs planar? Verify using Theorem 1.



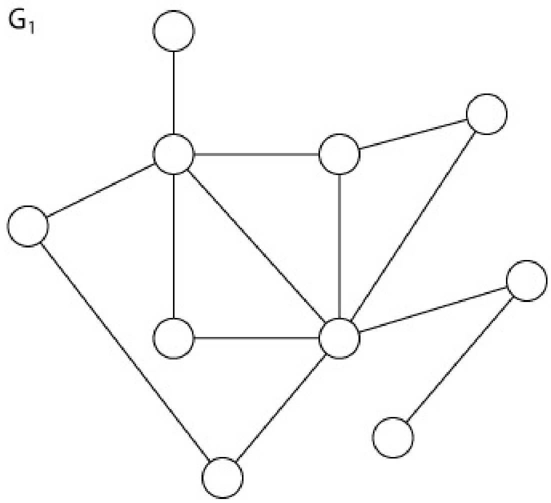


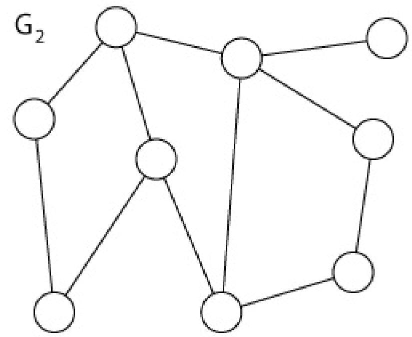
2. A **walk** is a sequence of vertices where each vertex is adjacent to the vertex before it and after it. A **path** is a walk that doesn't visit the same vertex twice. For each of the graphs in Problem 1, is a walk from s to y a path? Justify your answer.

3. Is the following graph bipartite? Why or why not? If you think the graph is bipartite, find the two groups of vertices A and B.



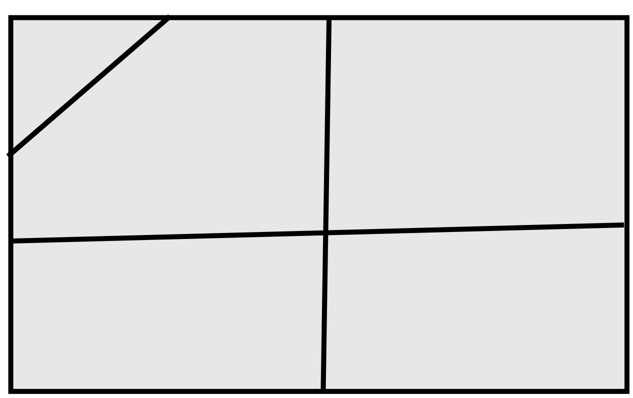
4. What is the fewest number of colours (i.e. chromatic number) required to colour the following graphs?





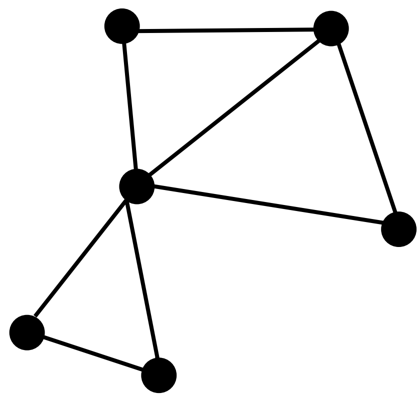
5. Graph theory can have some neat applications. Suppose you're given a map (like the one below). Find the minimum number of colours required to colour this map (no two adjacent regions can have the same colour). Note that regions intersecting at a point are **not** considered adjacent.

   Hint: Try representing each region of the map as a vertex in a graph. Adjacent regions will have an edge connecting them.



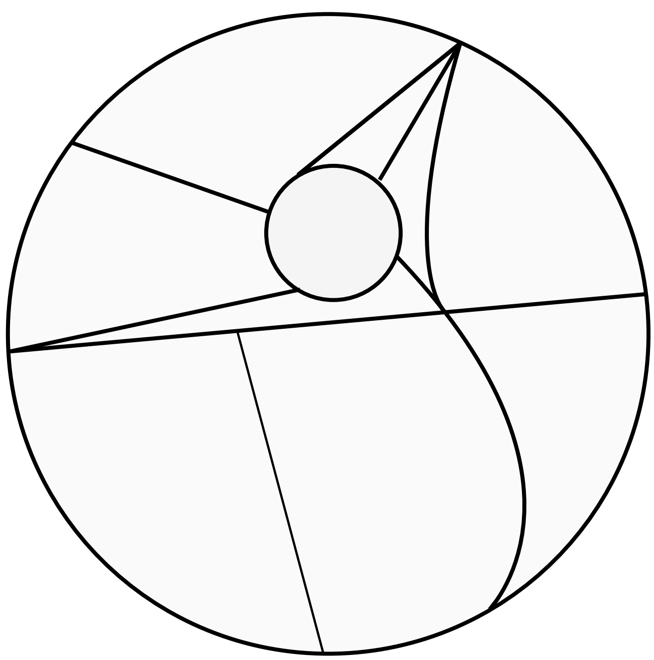
6. Is the graph you get from Problem 5 bipartite? Why or why not? If you think the graph is bipartite, find the two groups of vertices A and B.

7. Is the following graph bipartite? Why or why not? If you think the graph is bipartite, find the two groups of vertices A and B.



8. Find the chromatic number of the following map.

   Hint: The chromatic number is the least number of distinct colours needed to colour the map such that no two adjacent regions have the same colour.



9. Word has spread about your graph theory knowledge. The university has asked you to schedule a few of the first year courses for the upcoming term. There are 5 courses: Physics, Chemistry, Calculus, Circuits & Python. The possible course conflicts are shown in Table 1 below. Course conflicts are indicated with an 'X'.

Table 1: Course Conflicts

	Physics	Chemistry	Calculus	Circuits	Python
Physics		X	X	X	
Chemistry	X		X	X	X
Calculus	X	X		X	X
Circuits	X	X	X		
Python		X	X		

Each course is to be assigned a 1 hour time slot. Represent this situation as a graph and find the minimum number of time slots required to schedule all 5 courses without any conflicts.