Intermediate Math Circles
February 21, 2018
Contest Prep III

Problem Set

1. The symbol $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ equals $ad - bc$. If $\begin{array}{|c|c|}\hline x-1 & 2 \\ \hline 3 & -5 \\ \hline\end{array} = 9$, determine the value of x (2004 Pascal #18).

$$\begin{aligned}(x-1)(-5) - 2(3) &= 9\\ -5x + 5 - 6 &= 9\\ -5x - 1 &= 9\\ -5x &= 10\\ x &= -2\end{aligned}$$

2. John and Mary wrote the Cayley math contest. Two times John's score was 60 more than Mary's score. Two times Mary's score was 90 more than John's score. Determine the average of their two scores.

Solution 1:
Let J be John's score and M be Mary's score. Then $2J = M + 60$ and $2M = J + 90$.
Adding these two equations together, we get:

$$2J + 2M = M + J + 150 \implies J + M = 150 \implies \frac{J+M}{2} = 75$$

Therefore, the average of their two scores was 75.

Solution 2:
Let J be John's score and M be Mary's score.

Then $2J = M + 60 \implies M = 2J - 60$ (1)

We also have that $2M = J + 90$ (2)

Substituting (1) $\rightarrow$ (2):

$$\begin{aligned}2(2J-60) &= J + 90\\ 4J - 120 &= J + 90\\ 3J &= 210\\ J &= 70\end{aligned}$$

Substituting into (1) gives $M = 80$. Therefore, the average of their scores is 75.

3. In the diagram, each of the five boxes is to contain a number. For every box other than the two on the ends, the number in that box must be the average of the number in the box to the left of it and the number in the box to the right of it. Determine the values of x, y and z (2010 Pascal #21).

8	x	y	26	z

$$26 = \frac{y+z}{2} \implies y + z = 52$$

$$y = \frac{x+26}{2} \implies 2y = x + 26$$

$$x = \frac{8+y}{2} \implies 2x = 8 + y$$

$$\begin{array}{r} -x + 2y = 26 \\ 2x - y = 8 \end{array} \implies \begin{array}{c} -x + 2y = 26 \\ \underline{4x - 2y = 16} \\ 3x = 42 \\ x = 14 \end{array}$$

$$\therefore y = 2x - 8 = 2(14) - 8 = 20$$

$$\therefore z = 52 - y = 32$$

4. In each row of the table, the sum of the first two numbers equals the third number. Also, in each column of the table, the sum of the first two numbers equals the third number. What is the sum of the nine numbers in the table? (2006 Pascal #21)

m	4	$m+4$
8	n	$8+n$
$m+8$	$4+n$	6

From column 3:
$m + 4 + 8 + n = 6$
$m + n = -6$

From row 3:
$m + 8 + 4 + n = 6$
$m + n = -6$

The sum of the nine numbers in the table is:
$2(m+8) + 2(4+n) + (m+4) + (8+n) + 6$
$= 3m + 3n + 16 + 8 + 4 + 8 + 6$
$= 3(m+n) + 42$
$= 3(-6) + 42$
$= 24$

5. Suppose that x and y are positive numbers with

$$xy = \frac{1}{9}$$
$$x(y+1) = \frac{7}{9}$$
$$y(x+1) = \frac{5}{18}$$

What is the value of $(x+1)(y+1)$? (2011 Cayley #21)

Solution 1

Multiplying the 2^{nd} and 3^{rd} equations together we get:

$xy(x+1)(y+1) = \frac{7}{9} \cdot \frac{5}{18} = \frac{35}{162}$

but we know that $xy = \frac{1}{9}$, so:

$\frac{1}{9}(x+1)(y+1) = \frac{35}{162}$

$\therefore (x+1)(y+1) = \frac{35}{9 \cdot 18} \cdot 9 = \frac{35}{18}$

Solution 2

From 2^{nd} equation:

$xy + x = \frac{7}{9}$

$\frac{1}{9} + x = \frac{7}{9}$ since $xy = \frac{1}{9}$

$x = \frac{6}{9} = \frac{2}{3}$

From 3^{rd} equation:

$xy + y = \frac{5}{18}$

$\frac{1}{9} + x = \frac{7}{18}$

$y = \frac{5}{18} - \frac{2}{18} = \frac{3}{18} = \frac{1}{6}$

$\therefore (x+1)(y+1) = (\frac{2}{3}+1)(\frac{1}{6}+1) = \frac{5}{3} \cdot \frac{7}{6} = \frac{35}{18}$

6. Five positive integers are listed in increasing order. The difference between any two consecutive numbers in the list is three. The fifth number is a multiple of the first number. How many different such lists of five integers are there? (2007 Cayley #22)

Suppose the first number in the list is x.
Then the list would be: $x, x+3, x+6, x+9, x+12$.
$x + 12 = kx$, where k is an integer
$\therefore \frac{x+12}{x} = 1 + \frac{12}{x}$ is an integer
Therefore x is a positive divisor of 12. $\therefore x = 1, 2, 3, 4, 6,$ or 12
$\therefore$ There are 6 different lists.

7. Suppose that a, b and c are three numbers with

$$\begin{aligned} a + b &= 3 \\ ac + b &= 18 \\ bc + a &= 6 \end{aligned}$$

Determine the value of c (2009 Cayley #22).

Equations 2 + 3:

$ac + b + bc + a = 24$
$(a + b)c + (a + b) = 24$

since $a + b = 3$:

$3c + 3 = 24$
$3c = 21$
$c = 7$

8. There are four unequal, positive integers a, b, c and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$? (1998 Cayley #23)

$$\begin{array}{rl} N &= 5a + 3b + 5c \\ -N &= 4a + 5b + 4c \\ \hline 0 &= a - 2b + c \implies a + c = 2b \end{array}$$

$$\begin{aligned} \therefore N &= 4(a + c) + 5b \\ &= 8b + 5b \\ &= 13b\text{ , where } b \text{ is an integer} \end{aligned}$$

Multiples of 13 between 131 and 150:

$$\begin{aligned} 13 \times 10 &= 130 \\ 13 \times 11 &= 143 \leftarrow \text{the only one} \\ 13 \times 12 &= 156 \end{aligned}$$

$\therefore N = 143 \implies 13b = 143 \implies b = 11$

$$\therefore a + c = 2b = 22$$
$$\therefore a + b + c = 33$$

9. Solve the following system of equations (2000 Cayley #25 - modified)

$$x + xy + xy^2 = 26$$
$$x^2y + x^2y^2 + x^2y^3 = 156$$

$$x(1 + y + y^2) = 26 \qquad (1)$$
$$x^2y(1 + y + y^2) = 156 \qquad (2)$$

$x \neq 0$ since $0 \neq 26$
(2) $\div$ (1):

$$\frac{x^2y(1 + y + y^2)}{x(1 + y + y^2)} = \frac{156}{26}$$

$\therefore xy = 6 \implies y = 6/x$ and $x = 6/y$

Into (1):

$$\frac{6}{y}(1 + y + y^2) = 26$$

$$6 + 6y + 6y^2 = 26y$$

$$6y^2 - 20y + 6 = 0$$

$$\begin{aligned} y &= \frac{20 \pm \sqrt{20^2 - 4(6)(6)}}{2(6)} \\ &= \frac{20 \pm 16}{12} \\ &= 3\ ,\ {}^1/_3 \end{aligned}$$

$y = 3 \implies x = 2$
$y = {}^1/_3 \implies x = 18$
Therefore the solutions are $(x, y) = (2, 3), (18, \frac{1}{3})$