Intermediate Math Circles
February 7, 2018s
Contest Preparation I
Answers and Selected Solutions

Answers to Contest Preparation 1:

1. D	2. A	3. A	4. B	5. E	6. C
7. B	8. B	9. A	10. A	11. C	12. E
13. A	14. E	15. D	16. E	17. C	18. A
19. B	20. D	21. B	22. D	23. B	24. B
25. C					

Contest Preparation 1 Selected Solutions

4. Total Area $= 9 \times 7 = 63$
   Area of 4 identical triangles $= 4\left(\frac{2\times 2}{2}\right) = 8$
   Area of Shaded Region $= 63 - 8 = 55$ (B)

7. We know that negative numbers are smaller than positive numbers. The smallest number is obtained by multiplying the smallest negative number by the largest positive number.
   $\therefore (-9) \times 6 = -54$ is the smallest product. (B)

8. The prime factorization of $1995 = 3 \times 5 \times 7 \times 19$. Therefore the largest prime that divides 1995 is 19. (B)

11. We can check various powers of 3. There are only 4 possibilities. We check until we find the required power of 2.

$$\begin{aligned} 42 + 3^0 = 42 + 1 &= 43 \\ 40 + 3^1 = 40 + 3 &= 43 \\ 34 + 3^2 = 34 + 9 &= 43 \\ 16 + 3^3 = 2^4 + 3^3 &= 43 \end{aligned}$$

   $\therefore 2^4 + 3^3 = 2^x + 3^y$ and $x + y = 4 + 3 = 7$. (C)

12. Each coin is used the same number of times. The value of one of each coin is $0.01+0.05+0.10+0.25+1.00 = \$1.41$. The bank contains $\$5.64 = 4\times\$1.41$. There are four of each coin and there are five different coins. There are $5 \times 4 = 20$ coins in total. (E)

13.
$$\begin{aligned} & (12.5 \times 10^{-3}) \times (8 \times 10^{111}) \\ = & (12.5 \times 8) \times (10^{-3} \times 10^{111}) \\ = & 100 \times (10^{108}) \\ = & 10^2 \times (10^{108}) \\ = & 10^{110} \end{aligned}$$
(A)

14. 3 players sit out each game. After 6 games, 18 players sit out. That is, after six games each player has sat out once and has played 5 times. (E)

15. $EC = 7$ since it is a side in a $7-24-25$ right triangle. (You could calculate $EC = \sqrt{25^2 - 24^2} = \sqrt{625 - 576} = \sqrt{49} = 7$.) Since $ABCD$ is a rectangle, $AD = BC = 24$ and it follows that $DF = EC = 7$. Then $EF = 50 - 7 - 7 = 36$.
The perimeter of $ABEF = 50 + 25 + 36 + 25 = 136$. (D)

16. On the main diagonal, the numbers 4 and 5 have not been used. But there is already a 5 in the column containing Q. Therefore, $P = 5$ and the space immediately below the Q is 4. Now the column containing Q is missing a 2 and 3. But there is already a 3 in the row containing Q. Therefore, $Q = 2$ and $P + Q = 5 + 2 = 7$. (E)

17. Since 20 is even, 2 must be one of the three primes used. (The sum of three odd primes would be odd otherwise.) The primes less than 20 are 2, 3, 5, 7, 11, 13, 17 and 19. We can quickly discard 17 and 19 as possible numbers for use in the sum since all possible sums involving 17 or 19 are greater than 20. Only two combinations sum to 20: 2, 5, 13 and 2, 7, 11. (C)

18.
$$\begin{aligned} \frac{2}{5}x &= 16 - \frac{2}{5}x \\ \frac{4}{5}x &= 16 \\ 4x &= 80 \\ x &= 20 \end{aligned}$$

Now $\dfrac{5x}{2x+10} = \dfrac{5(20)}{2(20)+10} = \dfrac{100}{50} = 2$. (A)

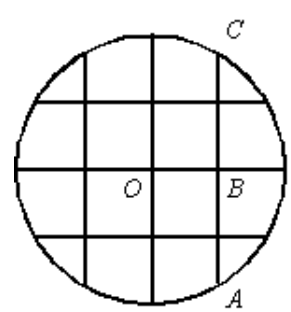
19. The cubes on the corners have three faces painted. The cubes on the edges but not the corners have two faces painted. The cubes on the faces but not the edges or corners have one side painted. Each edge, not includiong corners, contains 8 cubes and there are 12 edges. Therefore, $8 \times 12 = 96$ cubes have exactly two faces painted. (B)

20.
$$\begin{array}{rrcll}
 & 2w & = & 3x & (1)\\
 & w & = & y+z & (2)\\
 & x+y & = & z & (3)\\
\text{From (1)} & x & = & \frac{2}{3}w & (4)\\
\text{Substitue (4) in (3)} & z & = & \frac{2}{3}w+y & (5)\\
\text{Substitue (2) in (5)} & z & = & \frac{2}{3}(y+z)+y & \\
 & z & = & \frac{2}{3}y+\frac{2}{3}z+y & \\
 & 3z & = & 2y+2z+3y & \\
 & z & = & 5y &
\end{array}$$

Therefore it takes 5 y's to balance a z. (D)

21. The sum is $(1+4)+(1+4+9)+(1+4+9+16)+(1+4+9+16+25)+\cdots+(1+4+9+16+25+\cdots+n^2)$. The total must be less than or equal to 1000. So $5+14+30+55+91+140+204+285=824$ and $5+14+30+55+91+140+204+285+385=1209$. The closest sum under 1000 is 824. There would be $1000-824=176$ cubes remaining. (B)

22. The diagram shows the centre O, and two points on the circumference, A and C. Then OA and OC are radii such that $OC = OA = 2$. The point B is the midpoint of chord AC and is on a diameter so that $OB = 1$. Since the radius is 2, the diameter is 4. There are two diameters with a total lenth of $2 \times 4 = 8$. We know that $\triangle OBC$ is right angled so $CB = \sqrt{OC^2 - OB^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$. $CA = 2 \times CB = 2\sqrt{3}$. There are four chords with the same length as AC for a total length of $4 \times 2\sqrt{3} = 8\sqrt{3}$. The total length of the chords and diameters is $8\sqrt{3} + 8 = 8(\sqrt{3}+1)$. (D)



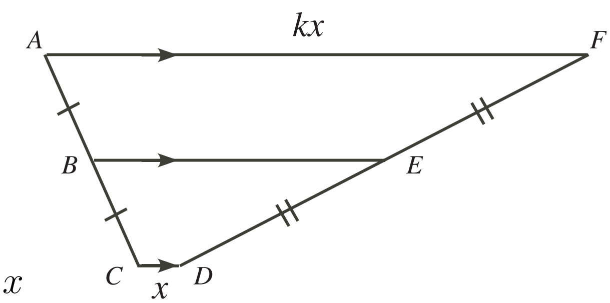
23. We know that $21xy^2 = (3)(7)xy^2$. For $21xy^2$ to be a perfect square, $x = (3)(7)$ and y can equal any number since y^2 is a perfect square. It follows that $15xy = 15(21)y = (3)(5)(7)(3)y = (3^2)(7)(5)y$. For this number to be a perfect square, $y = (5)(7)$. Therefore, the smallest value of $x+y = (3)(7)+(5)(7) = 56$. (B)

24. $2x + 1$ is an odd number for all x, where x is an integer. Let the 28 consecutive odd, positive integers be

$$2x-27, 2x-25, \cdots, 2x-3, 2x-1, 2x+1, 2x+3, \cdots, 2x+25, 2x+27$$

with a sum of $(2x)(28) = 56x = (2^3)(7)x$. We want this number to be a perfect cube. Therefore, $x = 7^2 = 49$. The smallest number is $(2x - 27)$ and the largest number is $(2x + 27)$. The average of these two numbers is $\dfrac{(2x-27)+(2x+27)}{2} = \dfrac{4x}{2} = 2x = 2(49) = 98$. (B)

25. Given: $AF : CD = k : 1$.



Let $AF = kx$ and $CD = x$, $x \neq 0$.

Area $ABEF$: Area $BCDE = 5 : 2$

Area of a trapezoid$= \frac{1}{2}(a+b)h$.

$BE =$ Average of AF and $CD = \dfrac{kx+x}{2}$.

$$\begin{aligned}
\frac{\text{Area } ABEF}{\text{Area } BCDE} &= \frac{\frac{1}{2}\left(kx + \frac{kx+x}{2}\right)h}{\frac{1}{2}\left(\frac{kx+x}{2} + x\right)h} \\
\frac{5}{2} &= \frac{\frac{2kx+kx+x}{2}}{\frac{kx+x+2x}{2}} \\
\frac{5}{2} &= \frac{3kx+x}{kx+3x} \\
5kx+15x &= 6kx+2x \\
13x &= kx \\
13 &= k, \quad x \neq 0
\end{aligned}$$

$\therefore k = 13$. (C)