Problem Set 1 - Solutions

Intermediate Math Circles Winter 2018
Fun With Inequalities

March 21, 2018

Linear Inequalities- Single Variable

Solve each of the following.

1. $x + 5 < \frac{7}{2}$

$$x < \frac{7}{2} - 5$$
$$x < \frac{7}{2} - \frac{10}{2}$$
$$x < \frac{-3}{2}$$

Therefore $x < \frac{-3}{2}$ satisfies the inequality.

2. $3 - \frac{x}{2} \geq -8$

$$-\frac{x}{2} \geq -8 - 3$$
$$-\frac{x}{2} \geq -11$$
$$-2(\frac{-x}{2}) \leq -2(-11)$$

We multiply by -2, so we flip the inequality.

$$x \leq 22$$

Therefore $x \leq 22$ satisifies the inequality.

3. $-1 - 3x \leq 4x + 10$

$$\begin{aligned} -3x - 4x &\leq 1 - +1 \\ -7x &\leq 11 \\ \frac{-7x}{-7} &\geq \frac{11}{-7} \end{aligned}$$

Since we are dividing by -7, we flip the inequality.

$$x \geq \frac{-11}{7}$$

Therefore $x \geq \frac{-11}{7}$ satisfies the inequality.

4. $2x + 5 > 4x - 7$

$$\begin{aligned} 5 + 7 &> 4x - 2x \\ 12 &> 2x \\ \frac{12}{2} &> \frac{2x}{2} \\ 6 &> x \\ x &< 6 \end{aligned}$$

Therefore $x < 6$ satisfies the inequality.

5. $-\frac{2}{3}x + \frac{3}{7} \leq 5 - \frac{x}{2}$

We clear the fractions by multiplying the LHS and the RHS by 42. We use 42 because it is the lowest common multiple.

$$\begin{aligned} 42(\frac{-2x}{3} + \frac{3}{7}) &\leq 42(5 - \frac{x}{2}) \\ -28x + 18 &\leq 210 - 21x \\ 18 - 210 &\leq -21x + 28x \\ -192 &\leq 7x \\ \frac{7x}{7} &\geq \frac{-192}{7} \\ x &\geq \frac{-192}{7} \end{aligned}$$

Therefore $x \geq \frac{-192}{7}$ satisfies the inequality.

Absolute Values

Solve each of the following algebraically. Check your answer graphically.

1. $|x+6| = 5$
   Case 1:$[(x+6) \geq 0]$

$$\begin{aligned} |x+6| &= 5 \\ (x+6) &= 5 \\ x &= 5-6 \\ x &= -1 \end{aligned}$$

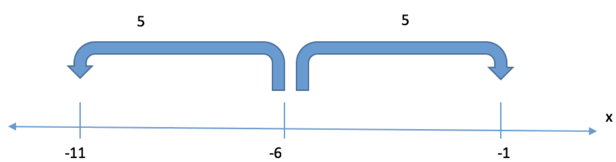
Case 2: $[(x+6) < 0]$

$$\begin{aligned} |x+6| &= 5 \\ -(x+6) &= 5 \\ -x-6 &= 5 \\ -x &= 11 \\ x &= -11 \end{aligned}$$

Therefore x = -1 and x = -11 satisfy the equation.

Check
Where is the special point?

$$\begin{aligned} x+6 &= 0 \\ x &= -6 \end{aligned}$$



2. $|x-4| \geq 1$
   Case 1: $[(x-4) \geq 0]$

$$\begin{aligned} |x-4| &\geq 1 \\ (x-4) &\geq 1 \\ x-4 &\geq 1 \\ x &\geq 5 \end{aligned}$$

Case 2: $[(x-4)<0]$

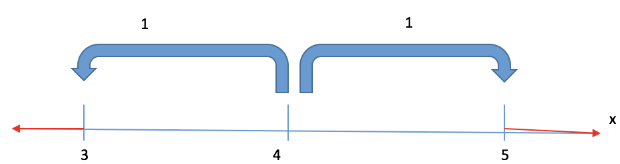
$$\begin{aligned} |x-4| &\geq 1 \\ -(x-4) &\geq 1 \\ -x+4 &\geq 1 \\ 4-1 &\geq x \\ 3 &\geq x \\ x &\leq 3 \end{aligned}$$

Therefore when $x \geq 5$ or when $x \leq 3$ the inequality is satisfied.

Check

Where is the "special" point?

$$\begin{aligned} x-4 &= 0 \\ x &= 4 \end{aligned}$$



3. $|4-x| \geq 1$

Notice $|4-x| = |-x+4| = |-1(x-4)|$

Question: Does |ab| equal |a| |b|?

We need to consider the four cases.

Case 1 [a $\geq$ 0 and b $\geq$ 0]: |ab|=ab=|a| |b|

Case 2 [a $\geq$ 0 and b $<$ 0]: We know ab $\leq$ 0 in this case.
This means that

$$\begin{aligned} |ab| &= -ab \\ &= a(-b) \\ &= |a||b| \end{aligned}$$

Case 3 [a < 0 and b ≥ 0]: Again we know ab ≤ 0.
This tells us that

$$\begin{aligned} |ab| &= -ab \\ &= (-a)(b) \\ &= |a||b| \end{aligned}$$

Case 4 [a < 0 and b < 0]: We know that ab > 0.
This means that

$$\begin{aligned} |ab| &= ab \\ &= (-a)(-b) \\ &= |a||b| \end{aligned}$$

Back to the observation that

$$\begin{aligned} |4-x| &= |-1(x-4)| \\ &= |-1||(x-4)| \\ &= 1|x-4| \\ &= |x-4| \end{aligned}$$

This means that question 3 is equivalent to question 2.

4. $|2x+1| < 7$

<u>Case 1 [(2x+1)≥0]:</u>

$$\begin{aligned} |2x+1| &< 7 \\ 2x+1 &< 7 \\ 2x &< 7-1 \\ \frac{2x}{2} &< \frac{6}{2} \\ x &< 3 \end{aligned}$$

<u>Case 2 [(2x+1)<0]:</u>

$$\begin{aligned} |2x+1| &< 7 \\ -(2x+1) &< 7 \\ -2x-1 &< 7 \\ -2x &< 8 \\ \frac{-2x}{-2} &> \frac{8}{2} \end{aligned}$$

We flip the inequality here!

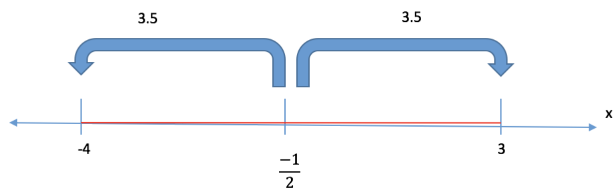
$$x > -4$$

Therefore the inequality holds for -4<x<3.

<u>Check</u>

Where is our special point?

$$2x + 1 = 0$$
$$x = \frac{-1}{2}$$



Wait! Why are we only going 3.5 from $\frac{-1}{2}$?
Because the x value is being doubled within this inequality.
In other words, we have 2x vs x.

5. $|x - 2| + |x + 5| = 8$

<u>Case 1 [(x-2) $\geq$ 0 and (x+5) $\geq$ 0]:</u>

$$|x - 2| + |x + 5| = 8$$
$$(x - 2) + (x + 5) = 8$$
$$2x + 3 = 8$$
$$2x = 5$$
$$x = \frac{5}{2}$$

<u>Case 2 [(x-2) $\geq$ 0 and (x+5) $<$ 0]:</u>

$$|x - 2| + |x + 5| = 8$$
$$x - 2 + [-(x + 5)] = 8$$
$$x - 2 - x - 5 = 8$$
$$-7 = 8$$

Case is inadmissable.

<u>Case 3 [(x-2) $<$ 0 and (x+5) $\geq$ 0]:</u>

$$
\begin{aligned}
|x-2|+|x+5| &= 8 \\
-(x-2)+(x+5) &= 8 - x + 2 + x + 5 \qquad = 8 \\
7 &= 8
\end{aligned}
$$

Case is inadmissable.

Case 4 [(x-2) < 0 and (x+5) < 0]:

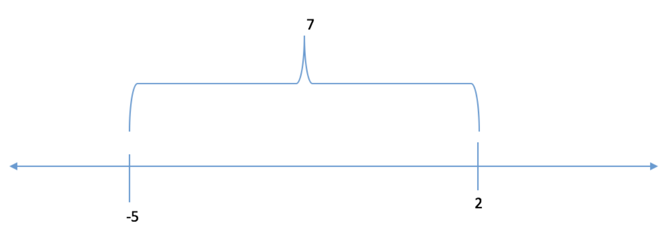
$$
\begin{aligned}
|x-2|+|x+5| &= 8 \\
-(x-2)+[-(x+5)] &= 8 \\
-x+2+(-x-5) &= 8 \\
-x+2-x-5 &= 8 \\
-2x-3 &= 8 \\
-2x &= 11 \\
x &= \frac{-11}{2}
\end{aligned}
$$

Therefore x = $\frac{5}{2}$ and x = $\frac{-11}{2}$ satisfy the equation.

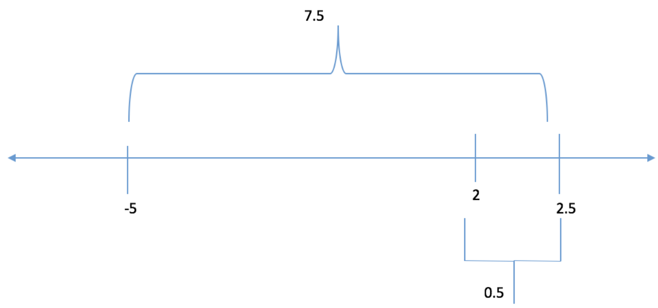
Check

Where are the special points?

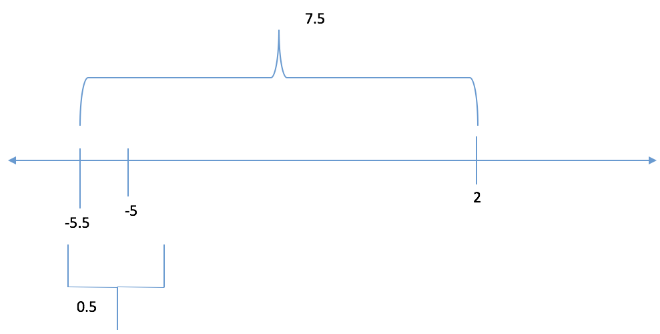
$$
\begin{aligned}
x-2 &= 0 \\
x &= -2 \\
x+5 &= 0 \\
x &= -5
\end{aligned}
$$



We know that for any x in the range -5 ≤ x ≤ 2 that |x-2| + |x+5| = 7.
To the right of 2 we need to add 1 so 7 + 1 = 8.



Similarly to the left of -5,



6. $|x| + |2 - x| \leq 12$

<u>Case 1 [x $\geq$ 0 and (2-x) $\geq$ 0]:</u>

$$\begin{aligned} |x| + |2 - x| &\leq 12 \\ x + 2 - x &\leq 12 \\ 2 &\leq 12 \end{aligned}$$

While this is true, it doesn't help us find the values that work.

<u>Case 2 [x $\geq$ 0 and (2-x) $<$ 0]:</u>

$$\begin{aligned} |x| + |2 - x| &\leq 12 \\ x + [-(2 - x)] &\leq 12 \\ x + (-2 + x) &\leq 12 \\ 2x - 2 &\leq 12 \\ 2x &\leq 14 \\ x &\leq 7 \end{aligned}$$

Case 3 [x < 0 and (2-x) ≥ 0]:

$$|x| + |2 - x| \leq 12$$
$$-x + 2 - x \leq 12$$
$$-2x + 2 \leq 12$$
$$-2x \leq 10$$
$$\frac{-2x}{-2} \geq \frac{10}{-2}$$

We flip the inequality.

$$x \geq -5$$

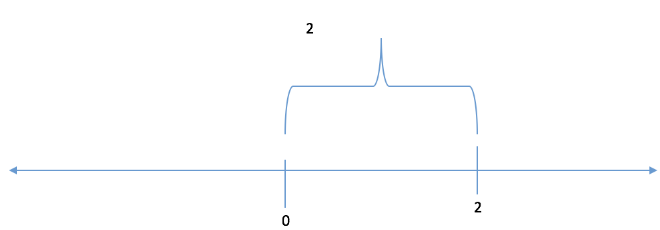
Case 4 [x < 0 and (2-x) < 0]:

$$|x| + |2 - x| \leq 12$$
$$-x + [-(2 - x)] \leq 12$$
$$-x - 2 + x \leq 12$$
$$-2 \leq 12$$

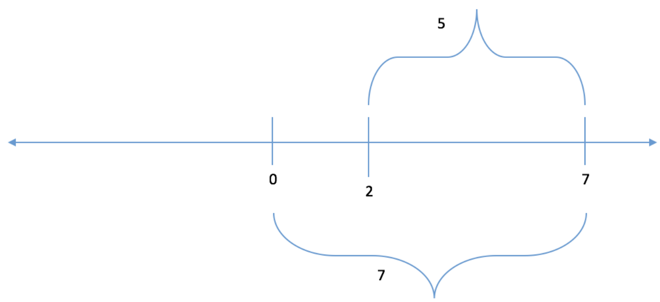
Again this is true, but doesn't help.
Therefore the inequality holds for -5 ≤ x ≤ 7.

Check
Where are the special points?

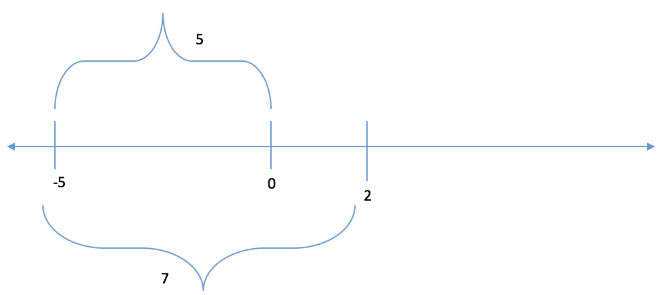
$$x = 0$$
$$2 - x = 0$$
$$2 = x$$
$$x = 2$$



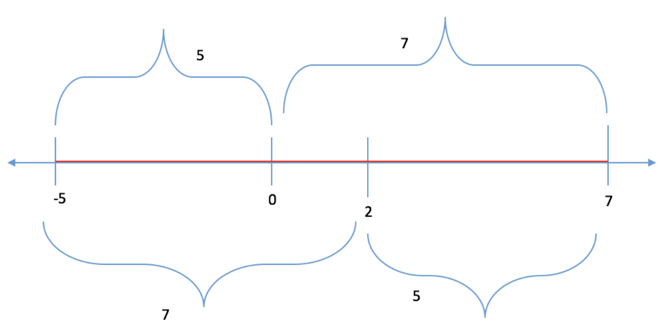
We know for any point in the range 0 < x < 2, |x| + |2-x|=2 and 2 ≤ 12.
To the right of 2 we can add 10 so that 2 + 10 = 12.



Similarly to the left of 0,



When we consider the last two number lines together, we get the following:



Properties

1. Which of the eight properties of $\leq$ also hold for $<$?

   The following properties hold for $<$ as well:
   (3) If $x < y$ and $y < z$, then $x < z$
   (4) One of the following three holds:
   $x < y$, $y < x$, or $x = y$
   (5) If $x < y$, then $x + r < y + r$
   (6) If $x < y$ and $r > 0$, then $rx < ry$
   (7) If $x < y$ and $4 < 0$, then $ry < rx$

2. Use whichever of the properties (1) to (8) that you need to prove the following

   (a) If $a \leq b$ and $c \leq d$, then $a + c \leq b + d$.

   Proof
   We know $a + c \leq b + c$ by property 5.
   We also know $b + c \leq b + d$ by property 5.
   Then by transitivity (property 3) we know $a + c \leq b + c \leq b + d$.
   Thus $a + c \leq b + d$.

   (b) If $0 \leq a \leq b$ and $0 \leq c \leq d$, then $0 \leq ac \leq bd$.

   Proof
   We know $a \leq b$ and $c \geq 0$.
   Then by property 6 we know $ca \leq cb$.
   Similarly, we know $bc \leq bd$ by property 6.
   Then by transitivity (property 3) we know $ca \leq cb = bc \leq bd$.
   Thus $ca \leq bd$.

3. (a) If $a \leq b$ and $c \leq d$, is it true that $ac \leq bd$?

   No! Consider the following counterexample:
   Let $a = -7$, $b = -1$, $c = -5$ and $d = 3$.
   Clearly $a \leq b$ and $c \leq$ d.
   But $ac = (-7)(-5) = 35$ and $bd = (-1)(3) = -3$.
   That's a problem because $bd \leq ac$ as $-3 \leq 35$.

   (b) If $a \leq b$, is it true that $\frac{1}{b} \leq \frac{1}{a}$?

   No! Consider the following counterexample:
   Let $a = -5$ and $b = -1$.

Thus $\frac{1}{a} = \frac{-1}{5}$ and $\frac{1}{b} = -1$.
But $-1 \leq \frac{-1}{5}$.
Note: the statement also hasn't dealt with the possibility of $a = 0$ or $b = 0$.

4. Show that if $a < b$, then $a < \dfrac{1}{2}(a+b) < b$.

   If $a < b$, we know by property 5 that

$$a + a < a + b$$
$$\frac{2a}{2} < \frac{a+b}{2}$$
$$a < \frac{a+b}{2}$$

   We also know by property 5

$$a + b < b + b$$
$$\frac{a+b}{2} < \frac{2b}{2}$$
$$\frac{a+b}{2} < b$$

   Then by transitivity (property 3) we know
   $a < \frac{a+b}{2} < b$.

5. Show that the sum of a positive number and its reciprocal is at least 2.
   In other words show that

$$a + \frac{1}{a} \geq 2$$

   We know by property 8 that

$$(a-1)^2 \geq 0$$
$$a^2 - 2a + 1 \geq 0$$
$$a^2 + \geq 2a$$

   Since $a > 0$ we can divide both the LHS and the RHS by a. That is

$$\frac{a^2+1}{a} \geq \frac{2a}{a}$$
$$\frac{a^2}{a} + \frac{1}{a} \geq 2$$
$$a + \frac{1}{a} \geq 2$$