Problem Set 2 Solutions

Intermediate Math Circles Winter 2018
More Fun With Inequalities

Rational Inequalities

1. $\frac{1}{x} \leq -7, x \neq 0$

   Solution:

   Case 1 $[x > 0]$

   $\frac{1}{x} \leq -7$

   $1 \leq -7x$

   $\frac{1}{-7} \geq \frac{-7x}{-7}$

   $x \leq \frac{-1}{7}$

   But $x > 0$ and $x \leq \frac{-1}{7}$ isn't possible.

   Thus this case is impossible.

   Case 2 $[x < 0]$

   $\frac{1}{x} \leq -7$

   $1 \geq -7x$

   $\frac{1}{-7} \leq \frac{-7x}{-7}$

   $x \geq \frac{-1}{7}$

   Thus $\frac{-1}{7} \leq x < 0$

2. $\dfrac{3}{x-2} \geq \dfrac{1}{4}, x \neq 2$

Solution:

Case 1 $[(x-2) > 0]$

$\dfrac{3}{x-2} \geq \dfrac{1}{4}$

$12 \geq x - 2$

$12 + 2 \geq x$

$x \leq 14$

If $x - 2 > 0$, we know $x > 2$ as well as $x \leq 14$

Thus, $2 < x \leq 14$

Case 2 $[(x-2) < 0]$

$\dfrac{3}{x-2} \geq \dfrac{1}{4}$

$12 \leq x - 2$

$12 + 2 \leq x$

$x \geq 14$

If $x - 2 < 0$, then $x < 2$. But there is no x such that $x < 2$ and $x \geq 14$.

This case is impossible

3. $\dfrac{x-3}{x+1} < 2, x \neq -1$

Solution:

Case 1 $[(x+1)>0]$

$\frac{x-3}{x+1}<2$

$x-3<2(x+1)$

$x-3<2x+2$

$-3-2<2x-x$

$-5<x$

If $x+1>0$, then $x>-1$ must also hold. Luckily if $x>-1$, then $x>-5$.

Thus, for this case $x>-1$

Case 2 $[(x+1)<0]$

$\frac{x-3}{x+1}<2$

$x-3>2(x+1)$

$x-3>2x+2$

$-3-2>2x-x$

$-5>x$

If $x+1<0$, then $x<-1$ must also hold. Luckily if $x<-5$, then $x<-1$.

Thus, for this case $x<-5$

Therefore, when $x>-1$ or when $x<-5$ the inequality is satisfied.

More Absolute Values

1. $|x|\geq 7$

   Solution:

   Case 1 $[x\geq 0]$

   $|x|\geq 7$

   $x\geq 7$

   Case 2 $[x<0]$

   $|x|\geq 7$

   $-x\geq 7$

   $x\leq -7$

   Therefore, when $x\geq 7$ or when $x\leq -7$ the inequality is satisfied.

2. $|x - 6| < 5$

Solution:

Case 1 $[(x - 6) \geq 0]$

$|x - 6| < 5$

$x - 6 < 5$

$x < 11$

If $x - 6 \geq 0$, then we know $x \geq 6$ as well as $x < 11$.

Thus, $6 \leq x < 11$.

Case 2 $[(x - 6) < 0]$

$|x - 6| < 5$

$-(x - 6) < 5$

$-x + 6 < 5$

$6 - 5 < x$

$x > 1$

If $x - 6 < 0$, then we know $x < 6$ as well as $x > 1$.

Thus, $1 < x < 6$.

Therefore, when we combine the two inequalities we get $1 < x < 11$
.

3. $|x + 2| \geq 8$

Solution:

Case 1 $[(x+2) \geq 0]$

$|x+2| \geq 8$

$x+2 \geq 8$

$x \geq 6$

If $(x+2) \geq 0$, then we know $x \geq -2$. Luckily if $x \geq 6$, then $x \geq -2$ as well.

Thus, $x \geq 6$

Case 2 $[(x+2) < 0]$

$|x+2| \geq 8$

$-(x+2) \geq 8$

$-x-2 \geq 8$

$-2-8 \geq x$

$-10 \geq x$

$x \leq -10$

If $(x+2) < 0$, then we know $x < -2$. Again if $x \leq -1$, then $x < -2$ as well.

Thus, $x \leq -10$

Therefore, when $x \geq 6$ or when $x \leq -10$ the inequality is satisfied.

4. $|3x| > 6$

Solution:
We know our "special" point is at zero with $|3x| > 6$ and we could use the argument we used in Problem Set 1 Question 4. That is, we need points $\frac{1}{3}$ of 6 away from our special point zero because we have $3x$ instead of x. Or we could use the fact that $|ab| = |a||b|$ discussed in Problem Set 1 Question 3 to show:

$|3x| > 6$

$|3||x| > 6$

$3|x| > 6$

$|x| > 2$

Therefore when $x < -2$ or $x > 2$ the inequality is satisfied.

5. $|x+1|+|x+6| \geq 4$

Solution:
Where are our "special" points?

$x+1=0$
$x=-1$

$x+6=0$
$x=-6$

Since the distance between the special points is 5, we know for all x on the number line $|x+1|+|x+6| \geq 5$

Since $5 \geq 4$ we know all values of x satisfy the inequality.

6. $|x-7|+|x-1| < 8$

Solution:

Where are our "special" points?

$x-7=0$
$x=7$

$x-1=0$
$x=1$

We know for $1 < x < 7$ that $|x-7|+|x-1| = 6$. To make sure $|x-7|+|x-1| < 8$ we can't be more than one away from 7 and one away from 1.

Therefore $0 < x < 8$ satisfy the inequality.

7. If a and b are any real numbers and $b \neq 0$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

Solution: In order to prove this we need to consider 4 cases.

Case 1 $[a \geq 0,\, b > 0]$

If $a \geq 0, b > 0$ and $\frac{a}{b} > 0$ we know $|\frac{a}{b}| = \frac{a}{b}$

Since $a \geq 0$ and $b > 0$ we know $|a| = a$ and $|b| = b$

Thus $|\frac{a}{b}| = \frac{a}{b} = \frac{|a|}{|b|}$

Case 2 $[a \geq 0, b < 0]$

If $a \geq 0, b < 0$ then $\frac{a}{b} \leq 0$ and $|\frac{a}{b}| = -\frac{a}{b}$

Since $a \geq 0$ and $b < 0$ we know $|a| = a$ and $|b| = -b$

Thus, $|\frac{a}{b}| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}$

Case 3 $[a < 0, b > 0]$

If $a < 0$ and $b > 0$, then $\frac{a}{b} < 0$ and $|\frac{a}{b}| = -\frac{a}{b}$

Since $a < 0$ and $b > 0$, we know $|a| = -a$ and $|b| = b$

Thus, $|\frac{a}{b}| = -\frac{a}{b} = \frac{-a}{b} = \frac{|a|}{|b|}$

Case 4 $[a < 0, b < 0]$

If $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$ and $|\frac{a}{b}| = \frac{a}{b}$

Since $a < 0$ and $b < 0$, we know $|a| = -a$ and $|b| = -b$

Thus, $|\frac{a}{b}| = \frac{a}{b} = \frac{-a}{-b} = \frac{|a|}{|b|}$

Therefore we know $|\frac{a}{b}| = \frac{|a|}{|b|}$ **when** a **and** b **are real numbers and** $b \neq 0$

8. If a is a real number and n is an integer, then $|a^n| = |a|^n$

Solution:

To prove $|a^n| = |a|^n$ we will consider the cases when $a \geq 0$ and $a < 0$

Proof:

Case 1 $[a \geq 0]$
If $a \geq 0$ then $a^n \geq 0$ and $|a| = a$

Thus $|a^n| = a^n = |a|^n$

Case 2 $[a < 0]$
If $a < 0$, then $|a| = -a$
If n is even, then $a^n > 0$ and $|a^n| = a^n$
With n even $(-1)^n = 1$
Thus $|a^n| = a^n = (-1)^n a^n = (-a)^n = |a|^n$

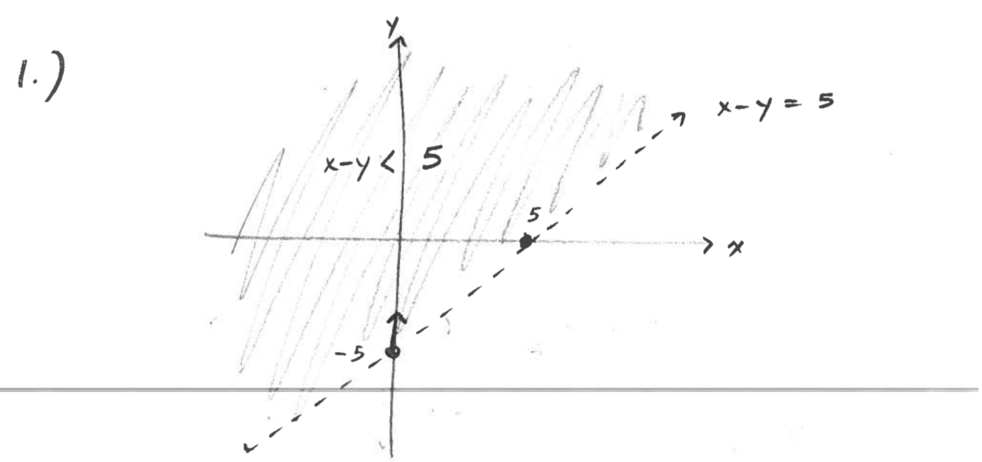
If n is odd, then $a^n < 0$ and $|a^n| = -a^n$
With n odd $(-1)^n = -1$
Thus $|a^n| = -a^n = (-1)a^n = (-1)^n a^n = (-a)^n = |a|^n$

Therefore we know $|a^n| = |a|^n$ when a is a real number and n is an integer

Two Variable Linear Inequalities

Graph the following regions that satisfy the inequalities

1. $x - y < 5$



Consider when $x = 0$ for $x - y < 5$

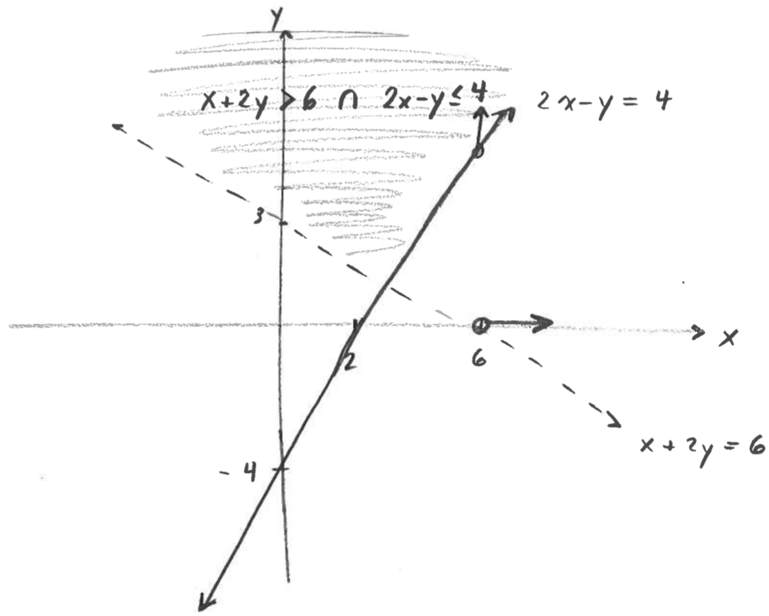
$0 - y < 5$

$-y < 5$

$y > -5$

2. $x + 2y > 6 \cap 2x - y \leq 4$

2.)



Consider $y = 0$ for $x + 2y > 6$,

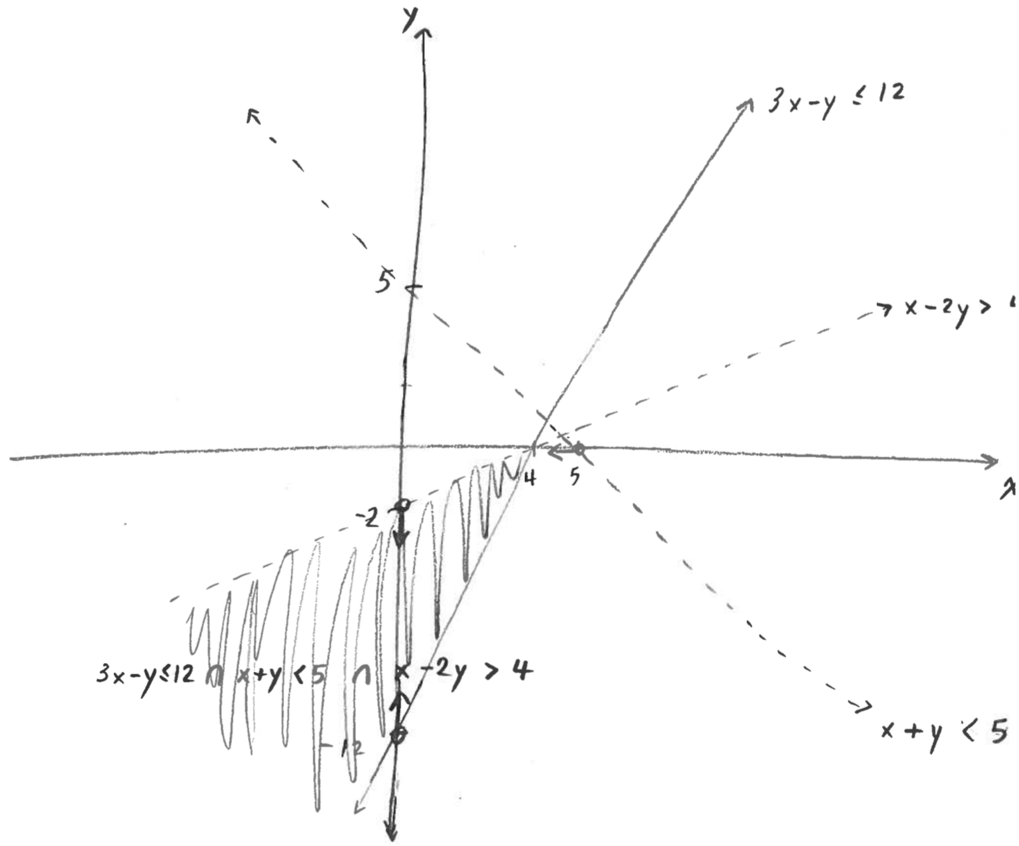
$$x + 2(0) > 6$$
$$x > 6$$

Consider $x = 6$ for $2x - y \leq 4$

$$2(6) - y \leq 4$$
$$12 - 4 \leq y$$
$$8 \leq y$$
$$y \geq 8$$

3. $3x - y \leq 12 \cap x + y < 5 \cap x - 2y > 4$

3.)



Consider $x = 0$ for $3x - y \leq 12$

$3(0) - y \leq 12$

$-12 \leq y$

$y \geq -12$

Consider $y = 0$ for $x + y < 5$

$x + 0 < 5$

$x < 5$

Consider $x = 0$ for $x - 2y > 4$

$0 - 2y > 4$

$-4 > 2y$

$y < -2$

4. Find a friend and try Inequality Battleship again. Except this time each of you uses a ship that only occupies a single lattice point.