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Centre for Education in
Mathematics and Computing

Grade 7/8 Math Circles

Winter 2018 - March 27/28/29

Word Problems - Solutions

Warm-up: The River Crossing Game

A family consisting of a mom, a dad, two boys and two girls, a policeman and a thief are on one side of a river and have only one boat to cross the river. Your goal is to help all 8 people cross the river safely while following these constraints:

1. The boat can carry no more than 2 people.
2. Only the parents or the policeman can drive the boat.
3. The dad can not be with the girls without the mom being there.
4. The mom can not be with the boys without the dad being there.
5. The thief can not be alone with any of the family members without the policeman being there.

Is it possible? If so, how many crossings will it take?

Retrieved from: https://play.google.com/store/apps/details?id=zl.puzzle.riveriq

1. Police Officer, Thief

2. Police Officer

3. Police Officer, Boy 1

4. Police Officer, Thief
5. Dad, Boy 2
6. Dad
7. Dad, Mom
8. Mom
9. Police Officer, Thief
10. Dad
11. Mom, Dad
12. Mom
13. Mom, Girl 1
14. Police Officer, Thief
15. Police Officer, Girl 2
16. Police Officer
17. Police Officer, Thief

Today we are going to work on word problems. These problems can often be solved in different ways. They will test your logic skills!

Tips and Tricks

To solve word problems, here is a step-by-step process you may find helpful for some scenarios, especially when you want to solve your equation algebraically!

1. Represent the unknown quantity with a variable.
2. Use the information given in the problem to set up an equation with the variable.
3. Solve the equation.
4. Write a conclusion.

Off to the races!

You have 25 horses and a track that can race 5 horses at once.
You are given no stopwatch and asked to find the 3 fastest horses.
You can assume that all the horses travel at different speeds and each horse travels at the same speed each race.
What is the minimum number of races required to determine the 3 fastest horses?
It takes 7 races to find the top 3. Let us consider the following matrix of horses, where left is fastest and right is slowest, and where each row is a different race.

1st	2nd	3rd	4th	5th
A	B	C	D	E
F	G	H	I	J
K	L	M	N	O
P	Q	R	S	T
U	V	W	X	Y

D, E, I, J, N, O, S, T, X and Y are not in the top of three of their races, so they are eliminated.
Next, we race all the winners:

1st	2nd	3rd	4th	5th
A	F	K	P	U

Now, P and U are eliminated since they are not in the top 3. Since P is better than Q and R, and since U is better than V and W, we eliminate Q, R, V and W.
Since K is at best third, we eliminate L and M.
Since F is at best second, we eliminate H, which is at best 4th.

We hold our last race, which races any runners-up (A is our winner):

1st	2nd	3rd	4th	5th
B	C	F	G	K

Therefore A, B and C are the top 3 horses.

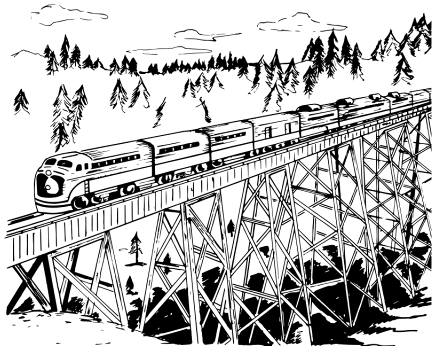
Watch out for that train!

A man is three eights of the way across a bridge when he hears a train coming from behind. If he runs as fast as possible back toward the train, he will get off the bridge just in time to avoid a collision.

Also, if he runs as fast as possible away from the train, he will get off the bridge (on the other side) just in time to avoid a collision.

The train is traveling at 40 kilometres per hour.

How fast does the man run?



<u>Solution 1</u>

The man can travel $\frac{3}{8}$ of the bridge to get to the start or $\frac{5}{8}$ of the bridge to get to the end. The train gets to the end of the bridge in the same amount of time that the man would take to run $\frac{5}{8}$ of the bridge. This distance is $\frac{5-3}{8} = \frac{1}{4}$ of the bridge.

He covers $\frac{1}{4}$ of the bridge in $\frac{40}{4} = 10$ kph.

<u>Solution 2</u>

Let x be the distance between the train and the bridge, let l be the length of the train, and let s be the speed of the man.

Recall that time $= \frac{distance}{speed}$. We know that $\frac{x}{40} = \frac{\frac{3l}{8}}{s}$. The train travels x km and at 40kph, while the man travels $\frac{3}{8}$ of the bridge at a speed of s.

We simplify this to get xs=15l, or l $= \frac{xs}{15}$.

Also, we know that $\frac{x+l}{40} = \frac{\frac{5l}{8}}{s}$. The first expression comes from the fact that the train travels to the end of the bridge, distance x + l, at a speed of 40 kph. The second expression comes from the fact that, in the same time, the man travels $\frac{5}{8}$ of the length of the bridge, at a speed s.

We simplify this to get that xs+sl = 25l

Or, xs + s($\frac{xs}{15}$) = 25($\frac{xs}{15}$)

Simplifying, we get that xs+$\frac{xs^2}{15} = \frac{25xs}{15}$

Bringing all terms to one side, we get that $\frac{15xs}{15} + \frac{xs^2}{15}$ - $\frac{25xs}{15} = 0$

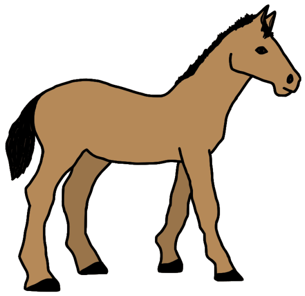
Factoring, we get that $\frac{xs}{15}$(15+s-25)=0

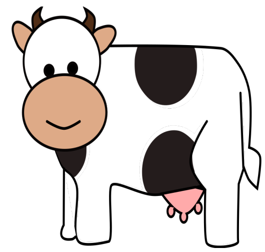
Our solutions are x = 0 (impossible), s = 0 (impossible), or s = 10.

Therefore the speed of the man is 10 kph.

Cow Many?

(CTMC 2016 Team problem number 20)





A group of cows and horses are randomly divided into two equal rows. (The animals are well-trained and stand very still.) Each animal in one row is directly opposite an animal in the other row. If 75 of the animals are horses and the number of cows opposite cows is 10 more than the number of horses opposite horses, determine the total number of animals in the group.

Solution 1

Let T be the total number of animals.

Suppose that, in the first line, there are C cows and H horses.

Suppose that, in the second line, there are w cows and x horses that are standing opposite the cows in the first line, and that there are y cows and z horses that are standing opposite the horses in the first line.

Since there are 75 horses, then $H + x + z = 75$.

Since there are 10 more cows opposite cows than horses opposite horses, then $w = z + 10$.

The total number of animals is $T = C + H + w + x + y + z$.

But $y + z = H$ and $C = w + x$ so $T = (w + x) + H + w + x + H = 2(H + w + x)$.

Now $w = z + 10$ so $T = 2(H + z + 10 + x) = 20 + 2(H + x + z) = 20 + 2(75) = 170$,

and so the total number of animals is 170.

Solution 2

Let h be the number of pairs consisting of a horse opposite a horse.

Let c be the number of pairs consisting of a cow opposite a cow.

Let x be the number of pairs consisting of a horse opposite a cow.

Since the total number of horses is 75, then $75 = 2h + x$.

(Each horse-horse pair includes two horses.)

Also, the given information tells us that $c = h + 10$.

The total number of animals is thus

2h + 2c + 2x = 2h + 2(h + 10) + 2(75 2h) = 2h + 2h + 20 + 150 4h = 170

Answer: 170

The Birthday Problem

In a group of 23 students, what is the probability that at least two people share the same birthday?

If there are two students, the probability they will have different birthdays is $\frac{365}{365} \times \frac{364}{365}$.

If there are three students, the probability they will have different birthdays is $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365}$.

If there are four students, the probability they will have different birthdays is $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365}$.

So, if there are 23 students, the probability they will have different birthdays is $\frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \frac{362}{365} \times \ldots \times \frac{343}{365}$.

Note: Another way we can write this is $\frac{364!}{342! \times 365^{22}} = 0.49$.

Then, we take the **complement** to find the probability that at least two people share a birthday.

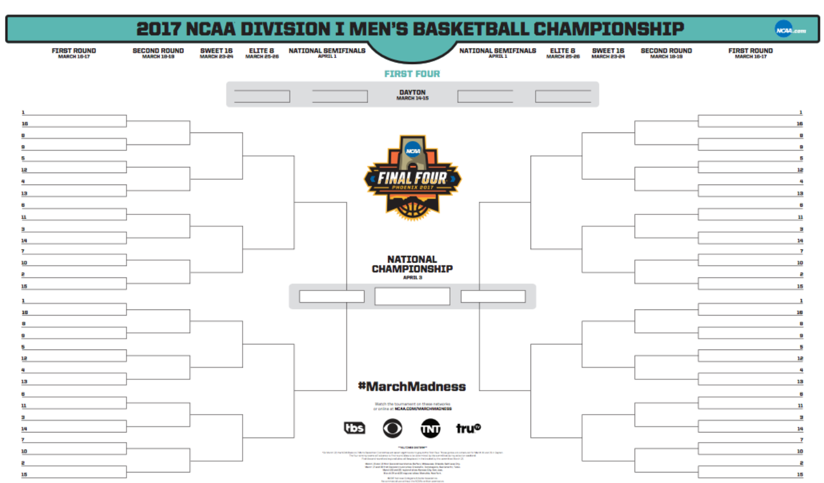
1 - 0.49 = 0.51

So the probability that at least two people share a birthday in a group of 23 students is 0.51.

March Madness!

Assuming there are 64 teams playing in the NCAA Basketball March Madness tournament and each team is equally likely to win any game (this part isn't true but otherwise the question will be much too hard!), what is the probability of filling out a perfect bracket?

A perfect bracket means that you guess the winner of all games correctly.

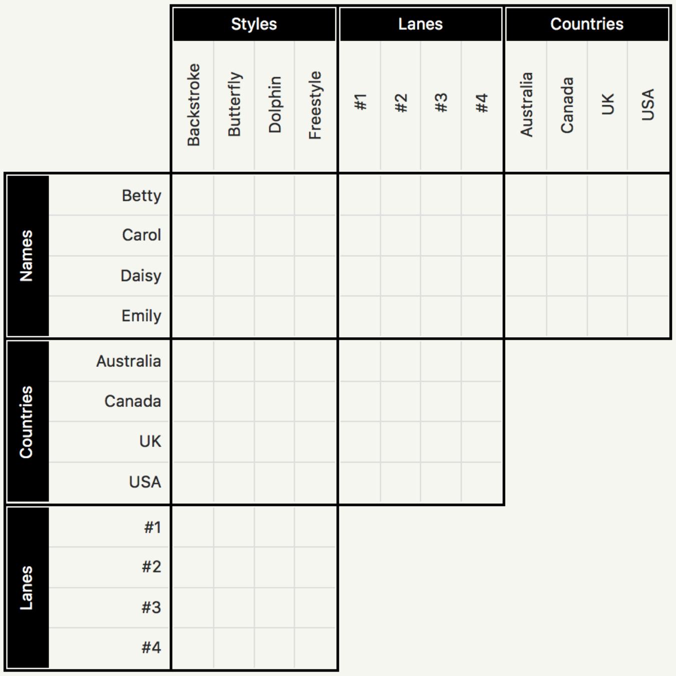


There are 64 teams, so there are 63 games played. This is because in a single elimination bracket tournament, every team except the winner loses exactly one game. For each game, you have a $\frac{1}{2}$ chance of guessing the winner correctly. So, for the entire tournament, you have 2^{63} different scenarios.

The probability of guessing all the winners correctly, or of filling out the perfect bracket, is $\frac{1}{2^{63}}$.

Swimming in Logic!

What stroke is being practiced by which swimmer from which country in which lane?



Clues Answer

1. Betty is swimming next to the athlete from the UK. Neither of them is swimming Butterfly.
2. Among Emily and the Backstroker, one is from the UK and the other is in the fourth lane.
3. Carol is not swimming Backstroke nor Dolphin. She is not Australian, and she is not swimming in lanes #2 nor #4.
4. The Freestyler is next to both Daisy and the American swimmer.
5. The American swimmer is next to Carol.
6. Daisy is not swimming in lane #2.

Retrieved from: https://www.brainzilla.com/logic/logic-grid/swimming-pool/

Names	Styles	Lanes	Countries
Betty	Dolphin	#2	USA
Carol	Butterfly	#1	Canada
Daisy	Backstroke	#4	Australia
Emily	Freestyle	#3	UK

The Monty Hall Problem

You are a contestant on a game show where the prize is a new car.

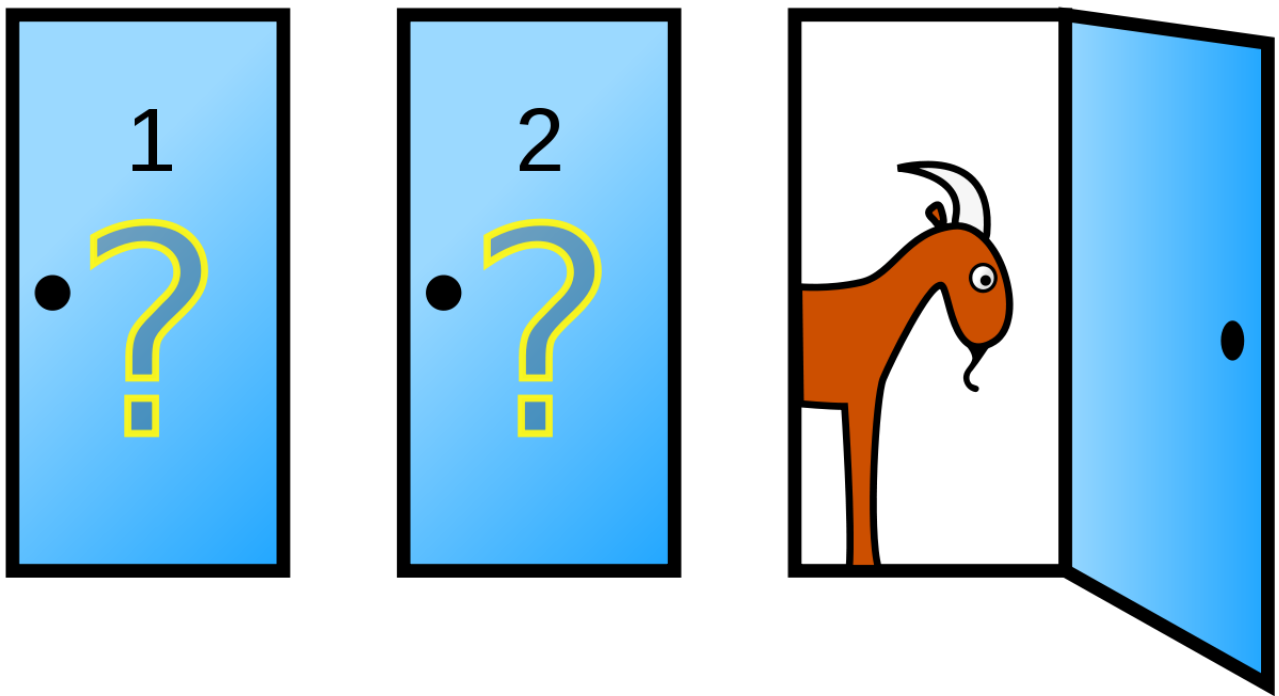
There are 3 doors. Behind two doors, there is a goat. Behind one door, there is a car.

You pick Door 1 but don't get to see what's behind any of the doors yet.

The host opens Door 3. There is a goat in this door!

Now, the host asks you whether you want to stick with Door 1 or switch to Door 2.

Which door do you choose?



You should switch doors! Here's why:

To begin, there is a 1 in 3 chance that you will pick the door with a car.

Now, you know that Door 3 has a goat. So, you "should" have a 1 in 2 chance of finding the car. It shouldn't matter which door you choose.

Think of it this way: If you stick with your door, you are still at 1 in 3 chances (from the beginning).

This means that together, doors 2 and 3 have a 2 in 3 chance that the car is behind them.

However, you have learned that the car is not behind Door 3. What this means is, all of the $\frac{2}{3}$ probability is in Door 2!

In other words, the probability of the car being behind doors 2 and 3 is Door 2 + Door 3 = $\frac{2}{3}$. But since the probability the car is behind Door 3 is 0, we have that Door 2 + Door 3 =

$\frac{2}{3} = \frac{2}{3} + 0$.

If this is difficult to understand, imagine you have 100 doors and you pick Door 1. You have a $\frac{1}{100}$ chance of winning. There was $\frac{99}{100}$ chance of the car being behind another door, and if the host opens 98 other doors, then the probability concentrates around the one door not chosen by you that has not been opened.

Problems

1. A large box of chocolates and a small box of chocolates together costs \$15. If the large box costs twice as much as the small box, what are the individual prices of the two boxes? (Gauss 2007).

Knowns:	Unknowns:
Total cost = \$15	Let the price of the large box in dollars be L.
	Let the price of the small box in dollars be S.

Equations:

From the question: the total cost results from buying one large box and one small box:

$$L + S = 15$$

The big box is twice the size of the small box, so we could buy two small boxes for the price of a big box:

$$L = 2 \times S$$

Now we have a system of equations:

$$L + S = 15 \tag{1}$$
$$L = 2 \times S \tag{2}$$

Solve one of the equations for any variable and ¨plug" in the expression on the other side into a different equation. Then, solve the new equation. This is called the substitution method and we can use it to solve this system of equations to get the value of our two unknowns. We can see in our system that equation (2) is already solved for the variable L. We plug in to expression on the other side of equation (2) into L in equation (1):

$$\begin{aligned} L + S &= 15 \\ (2 \times S) + S &= 15 \\ 3 \times S &= 15 \\ S &= 5 \end{aligned}$$

I used brackets when I plugged in L (second step) to keep track of my substitution and to preserve BEDMAS. Always do this!

I now plug $S = 5$ back into equation (2) to solve for L:

$$L = 2 \times S$$
$$L = 2 \times 5$$
$$L = 10$$

Therefore, the price of the large box is \$10 and the price of the small box is \$5.

2. The sum of three consecutive odd numbers is 57. What is the product of these three numbers? (Gauss 2014)

Knowns:	**Unknowns:**
Sum $= 57$	Let the smallest number be S.
	Let the middle number be M.
	Let the largest number be L.

Equations:

Let's do the easy one first:

$$S + M + L = 75$$

We usually denote an even number like this: $2n$, and an odd number like this: $2n + 1$ where n is any number greater than or equal to zero. So let's say:

$$S = 2n + 1$$
$$M = 2n + 3$$
$$L = 2n + 5$$

Notice that I could come up with these formulas because I knew that consecutive odd numbers are two apart.

Now, I'm going to plug in my expressions for S, M, and L into my sum equation to relate it to my known.

$$\begin{aligned} S + M + L &= 57 \\ (2n + 1) + (2n + 3) + (2n + 5) &= 57 \\ 6n + 9 &= 57 \\ 6n &= 48 \\ n &= 8 \end{aligned}$$

I can plug this value for n back into my equations for S, M, and L to solve for the three unknowns.

$$S = 2n + 1 = 2(8) + 1 = 17$$
$$M = 2n + 3 = 2(8) + 3 = 19$$
$$L = 2n + 5 = 2(8) + 5 = 21$$

My three consecutive odd numbers are 17, 19, and 21. I have to find their product:

$$17 \times 19 \times 21 = 6783$$

Therefore, the product of the three numbers is 6783.

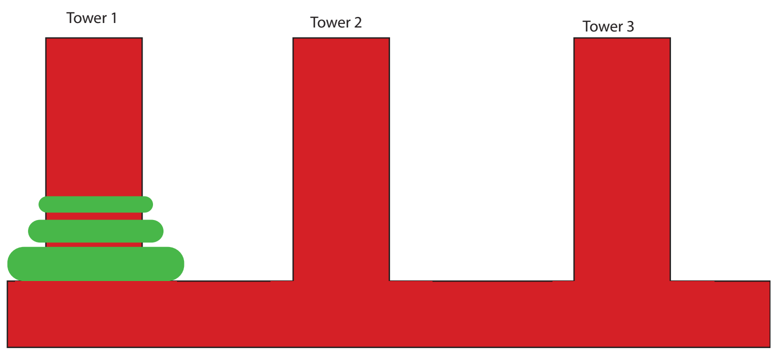
3. Four basketball teams from the Waterloo, Laurier, Queens and Toronto compete in the University March Madness Basketball Tournament. The tournament is single elimination with the first round match ups Waterloo vs. Laurier and Queens vs. Toronto. If someone was to randomly guess the winners for the entire tournament, what is the probability they would guess correctly?
There are 4 teams in the tournament, so there are 3 games in a single elimination tournament. Therefore, the probability you would guess correctly is $\frac{1}{2^3} = \frac{1}{8}$.

4. In football, the player who kicks the ball is referred to as the punter. During a recent football game, the punter, Khan Kickit, kicked the ball five times. His longest kick was 44 yards and he averaged 35 yards per kick. Each of his kicks was a different positive integer length.
Determine the minimum possible length of Khan's shortest kick.

We have that the average of 44 and the other kicks, represented here by w, x, y and z, is 35 yards.
That is, $\frac{44+w+x+y+z}{5} = 35$.
Or, 44 + w + x + y + z = 175.
131 = w + x + y + z
Assuming we have the largest possible values for 4 of the 5 kicks, we have 131 = 43 + 42 + 41 + x
Or, 131 = 126 + x
So, x = 5.

Therefore, the minimum possible length of Khan's shortest kick is 5 yards.

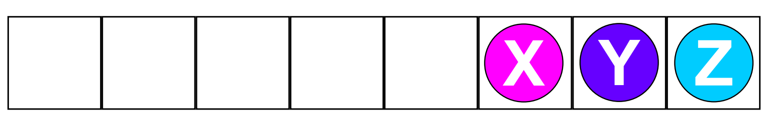
5. The Tower of Hanoi is a game with three towers, and a certain amount of rings. The object of the game is to move all of the rings from the first tower to the third tower in the same order (largest at the bottom to smallest at the top). The catch: You can only move one peg at a time, and a ring can never have a larger ring on top of it.



What is the minimum number of moves for 3 rings? 5 rings?

For 3 rings, the minimum number of moves is 7 and for 5 rings, it is 31 moves.

6. Three counters, labelled X, Y, and Z are placed in the positions shown on the board. Two players alternate turns. A turn consists of moving any one of the three counters any number of squares to the left, but the counter may not land on top of, or move past any of the other counters. The winner of the game is the player who makes the last legal move. What is the winning strategy? Is the player going first or the player going second guaranteed to win with this strategy?



The player going first can guarantee that they will win the game if they always move the counters the farthest possible to the left each move.

7. A census taker approaches a woman leaning on her gate and asks about her children. She says, "I have three children and the product of their ages is thirty-six. The sum of their ages is the number on this gate." The census taker does some calculation and claims not to

have enough information. The woman enters her house, but before slamming the door tells the census taker, "I have to see to my eldest child who is in bed with measles." The census taker departs, satisfied.

What are the ages of the three children?
Even knowing the number on the gate, the census taker could still not figure out the ages of the children.
That means there were two possibilities: 9,2,2 and 6,6,1.
However, there was only one oldest child, so they must be 9,2, and 2.

8. Five sisters all have their birthday in a different month and each on a different day of the week. Using the clues below and the grid, determine the month and day of the week each sister's birthday falls.

1. Paula was born in March but not on Saturday. Abigail's birthday was not on Friday or Wednesday.
2. The girl whose birthday is on Monday was born earlier in the year than Brenda and Mary.
3. Tara wasn't born in February and her birthday was on the weekend.
4. Mary was not born in December nor was her birthday on a weekday. The girl whose birthday was in June was born on Sunday.
5. Tara was born before Brenda, whose birthday wasn't on Friday. Mary wasn't born in July.

(Source: puzzlersparadise.com)

	February	March	June	July	December	Sunday	Monday	Wednesday	Friday	Saturday
Abigail										
Brenda										
Mary										
Paula										
Tara										
Sunday										
Monday										
Wednesday										
Friday										
Saturday										

Abigail - February - Monday
Brenda - December - Wednesday
Mary - June - Sunday
Paula - March - Friday
Tara - July - Saturday

9. At Cornthwaite H.S., many students enroll in an after-school arts program. The program offers a drama class and a music class. Each student enrolled in the program is in one class or both classes. In 2014, a total of 80 students enrolled in the program. Let x represent the number of students in both classes. If there were five fewer than thrice x students in the drama class, and thirteen greater than six times x students in the music class, how many students were in both classes?

The number of students in the drama class is $3x - 5$ and the number of students in the music class is $6x + 13$. Together, the number of students in both classes should add up to the total number of students enrolled, 80. However, the number of students in both classes appear in both the equations above. In order to prevent double counting them, we have to subtract the total number of students in both classes, x, from one of the equations above. Thus:

(3x - 5 - x) + (6x + 13) = 80

(2x - 5) + (6x + 13) = 80

8x + 8 = 80

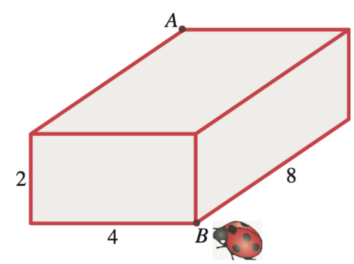
8x = 72

x = 9

Therefore, there are 9 students enrolled in both classes.

10. A ladybug wishes to travel from B to A on the surface of a wooden block with dimensions $2 \times 4 \times 8$ as shown in the diagram.

Determine the shortest distance for the ladybug to walk.



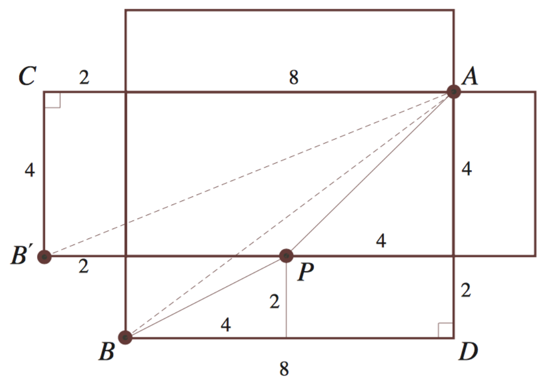
<u>Solution</u>

Many strategies could be attempted. Perhaps the bug walks along the edges and travels $8+2+4=14$ units. Perhaps the bug travels across the right side of the block from B to the midpoint of the top edge (marked P on the diagram below) and then across the top of the box to A. Referring to the diagram below, it can be shown that this distance is

$$BP+PA=\sqrt{BP^2}+\sqrt{PA^2}=\sqrt{4^2+2^2}+\sqrt{4^2+4^2}=\sqrt{20}+\sqrt{32}\doteq 10.13 \text{ units.}$$

But is this the shortest distance?

To visualize the possible routes fold out the sides of the box so that they are laying on the same plane as the top of the box. Label the diagram as shown below. Note that as a result of folding out the sides, corner B appears twice. The second corner is labelled B'.



The shortest distance for the ladybug to travel is a straight line from B to A or B' to A. So both cases must be considered.

BA is the hypotenuse of right-angled triangle ABD. Using Pythagoras' Theorem,

$$BA^2=BD^2+DA^2=8^2+6^2=100 \text{ and } BA=10 \text{ follows.}$$

$B'A$ is the hypotenuse of right-angled triangle $AB'C$. Using Pythagoras' Theorem,

$$(B'A)^2=(B'C)^2+CA^2=4^2+10^2=116 \text{ and } B'A=2\sqrt{29}\doteq 10.77 \text{ follows.}$$

Since $BA<B'A$, the shortest distance for the ladybug to travel is 10 units on the surface of the block in a straight line from B to A.

This problem is quite straight forward once the three-dimensional nature of the problem is removed.