



Grade 7/8 Math Circles
Winter 2018 - March 6/7/8
The Mathematics of Voting - Solutions

Warm-up: Time to vote!

We need to decide what the class preference is: candy, chocolate or chips. Think about your answer and we will take a vote!

Record the number of votes received for each food. The food with the most votes wins!

Candy	Chocolate	Chips

Now we are going to do the same vote but in a different way. You are going to be split up based on your birth month as follows:

- Group A: January, February, March
- Group B: April, May, June
- Group C: July, August, September
- Group D: October, November, December

The food preferred by the most groups wins in this case.

Definitions

- A **candidate** is a person or thing that we can vote for in an election
- A **voter** is someone who votes in an election
- An **election** is a process where a population votes to make a decision
- A **ballot** is a device used to express the voters' opinions of the candidates
- An **outcome** is a result of an election

Here are some types of ballots:

- A **single-choice ballot** asks voters to choose the one candidate that they prefer. An example is our federal election.
- A **preference ballot** asks voters to rank all candidates in their order of preference. An example is if I asked you to rank all 3 of candy, chocolate and chips.
- A **truncated preference ballot** asks voters to rank their first few preferred candidates. An example is when you vote for players in the NHL All-Star game.

Here are some types of outcomes from elections:

- A **winner-only** outcome indicates the winner of the election. An example is the Grammy awards.
- A **partial ranking** ranks the first few candidates by number of votes. An example is a math club where first place is the president and second place is the vice-president.
- A **full ranking** ranks all candidates by number of votes. An example is a reality show like *American Idol*.

Finally, we look at a **preference schedule**, which tells us how many voters voted for a certain order of candidates. An example is shown below:

Consider our warm-up example with candy, chocolate and chips. How many different ballots are possible?

$3! = 6$. This is a permutation (seen in probability). $3! = 3 \times 2 \times 1 = 6$.

Number of voters	15	20	13	10	19	8
1	Chocolate	Chocolate	Candy	Candy	Chips	Chips
2	Candy	Chips	Chips	Chocolate	Chocolate	Candy
3	Chips	Candy	Chocolate	Chips	Candy	Chocolate

This means that 15 people voted for the order "chocolate, candy, chips", 20 people voted for the order "chocolate, chips, candy" and so on.

In total, how many people voted for each food as their favourite?

Chocolate: 35; Candy: 23; Chips: 27

Methods of Voting

1. Plurality

The plurality method counts only the number of first-placed votes. The candidate with the highest number of first-placed votes wins the election.

Example

Evan (E), Hannah (H), Simon (S) and Annie (A) are running for school president. Their school has a ranked election. Here are the results:

Number of voters	14	10	8	4	1
1st	A	S	E	H	S
2nd	H	H	S	E	E
3rd	S	E	H	S	H
4th	E	A	A	A	A

Who wins the election based on the plurality method?

Annie has 14 votes and wins the election

If the school gives 2nd place the position of vice-president, 3rd place the position of treasurer and 4th place the position of secretary, who fills each position?

Annie wins with 14 votes, Simon comes second with 11 votes, Evan comes third with 8 votes, Hannah comes 4th with 4 votes. Therefore, Annie is president, Simon is vice-president, Evan is treasurer and Hannah is secretary.

2. Borda Count

The Borda count method gives points in increasing order based on number of votes. Often, last place will get 1 point, second last place will get 2 points, all the way up until first place, who gets the maximum number of points for a vote. The candidate with the most points is the winner.

Example

Returning to the school presidential election, we will allocate 4 points to a first place vote, 3 points for a second place vote, 2 points for a third place vote and 1 point for a fourth place vote. Here is the preference schedule:

Number of voters	14	10	8	4	1
1st (4 pts)	A (56)	S (40)	E (32)	H (16)	S (4)
2nd (3 pts)	H (42)	H (30)	S (24)	E (12)	E (3)
3rd (2 pts)	S (28)	E (20)	H (16)	S (8)	H (2)
4th (1 pt)	E (14)	A (10)	A (8)	A (4)	A (1)

Using the Borda count method, who wins the election?

Annie has $56 + 10 + 8 + 4 + 1 = 79$ points

Hannah has $42 + 30 + 16 + 16 + 2 = 106$ points

Simon has $28 + 40 + 24 + 8 + 4 = 104$ points

Evan has $14 + 20 + 32 + 12 + 3 = 81$ points

Therefore, Hannah wins the election.

What is the maximum possible points a candidate can get?

If a candidate gets all 37 first place votes, they will have $37 \times 4 = 148$ points.

3. Plurality with elimination

The plurality with elimination method is a multi-step process. After a ranked election, the candidate with the fewest first-place votes is eliminated. Their votes are transferred to the voter's second ranked candidate (or next-highest ranked candidate who has not been eliminated). This process is repeated until one candidate has a majority, at which point they are declared the winner.

Example

Recall the school presidential election with Evan, Hannah, Simon and Annie. Here is the preference schedule:

Number of voters	14	10	8	4	1
1st	A	S	E	H	S
2nd	H	H	S	E	E
3rd	S	E	H	S	H
4th	E	A	A	A	A

Round 1

Here is a summary of the number of votes for each candidate:

Candidate	A	H	S	E
Votes	14	4	11	8

Hannah is eliminated (she has the fewest votes). Her votes are transferred to Evan, since everyone who made Hannah their first-place vote also made Evan their second-place vote.

Round 2

Candidate	A	S	E
Votes	14	11	12

Simon is eliminated with the lowest number of votes (11). From the preference schedule, we see that 10 of his votes would go to Hannah, but she has been eliminated! So, 10 of Simon's votes go to Evan. And the other vote goes Evan as well.

Round 3

Candidate	A	E
Votes	14	23

Evan has 23 votes. Since there were a total of 37 votes, Evan has a majority and wins the election!

Practice

We are down to the top 5 of *American Idol*! Below is a preference schedule for the remaining contestants. Who wins the show, using the plurality with elimination method?

Number of voters	93	44	10	30	42	81
1st	Adam	Blake	Carrie	Carrie	David	Elise
2nd	Blake	David	Adam	Elise	Carrie	David
3rd	Carrie	Elise	Elise	Blake	Elise	Carrie
4th	David	Carrie	Blake	Adam	Adam	Blake
5th	Elise	Adam	David	David	Blake	Adam

Round 1

Candidate	Adam	Blake	Carrie	David	Elise
Votes	93	44	40	42	81

Carrie has the least number of first-place votes and is eliminated first. 10 of her votes go to Adam, 30 go to Elise.

Round 2

Candidate	Adam	Blake	David	Elise
Votes	103	44	42	111

David has the fewest first-place votes, so he is eliminated next. All 42 of his votes would go to Carrie, but she has already been eliminated, so they will go to Elise.

Round 3

Candidate	Adam	Blake	Elise
Votes	103	44	153

Elise now has 153 of 300 possible votes. She has the majority, so we do not need to vote again. Elise is your *American Idol*!

4. Pairwise Comparison

The pairwise comparison method puts candidates in a head-to-head setting. Based on the preference schedule, 1 point is given to a candidate that is ranked higher by the most voters, 0 points are given to the candidate ranked lower, and $\frac{1}{2}$ a point is given to both candidates if it is a tie.

Example

We use the same example, with the preference schedule given below:

Number of voters	14	10	8	4	1
1st	A	S	E	H	S
2nd	H	H	S	E	E
3rd	S	E	H	S	H
4th	E	A	A	A	A

We need to check 6 different pairs: A vs. H, A vs. S, A vs. E, H vs. S, H vs. E, and S vs. E.

Let's see Annie vs. Hannah: Annie has 14 votes where ranks above Hannah, while Hannah has the other 23. Therefore, Hannah beats Annie in this pairwise comparison and gains 1 point.

The table below can help to show pairwise comparisons. Fill in the rest:

Pairwise comparison	Votes	Winner
A vs. H	A (14); H (23)	H
A vs. S	A (14); S (23)	S
A vs. E	A (14); E (23)	E
H vs. S	H (18); S (19)	S
H vs. E	H (28); E (9)	H
S vs. E	S (25); E (12)	S

Who has the most total points (and is therefore the winner)?

Hannah has 2 points, Simon has 3 points, Evan has 1 point, Annie has 0 points. Therefore Simon is the winner.

For an election of N candidates, how many pairwise comparisons will need to be done?

We have 6 comparisons for 4 candidates. For N candidates, we have $\frac{N(N-1)}{2}$

Which method is the best?

Here is a summary of the school presidential election results using the 4 voting methods:

Method	1st	2nd	3rd	4th
Plurality	A	S	E	H
Borda count	H	S	E	A
Plurality with elimination	E	A	S	H
Pairwise comparison	S	H	E	A

What do you notice about these results?

Someone different wins for each voting method!

Arrow's Impossibility Theorem

Given three or more candidates, it is mathematically impossible to have a fair voting method.

What does fair mean?

Here are the 4 **fairness criteria**:

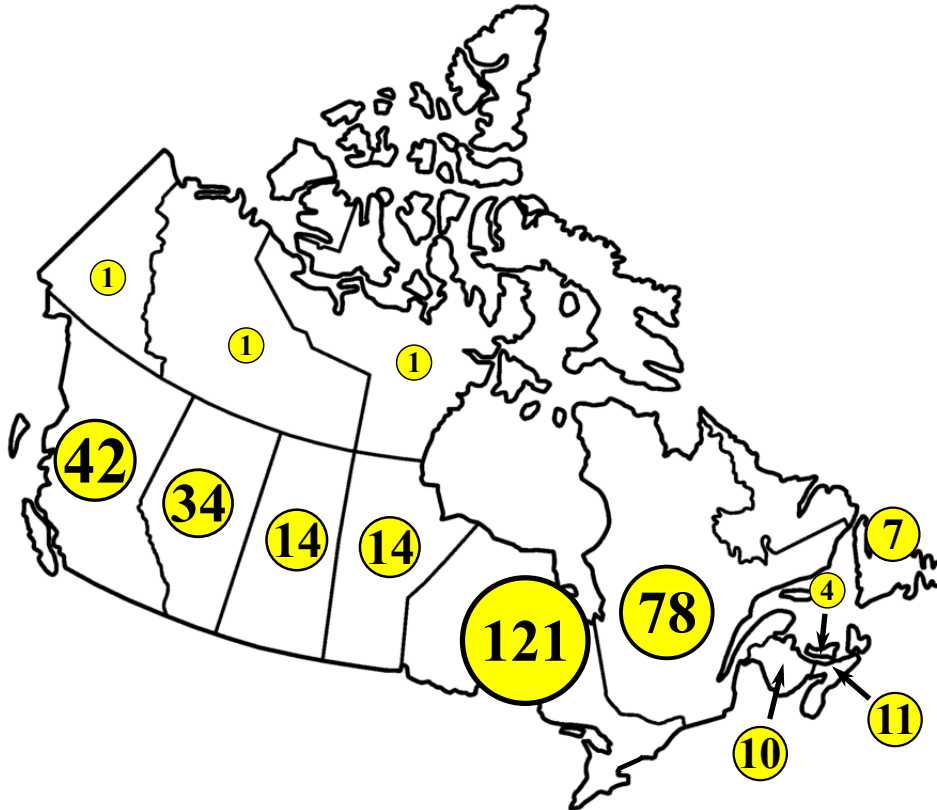
1. Majority: The candidate with the majority of first-place votes should win
2. Condorcet: The candidate who is preferred to the other candidates in pairwise comparison should win
3. Monotonicity: A candidate should not be negatively affected by a voter moving them up higher on their ballot
4. Independence of Irrelevant Alternatives: The winner should not be changed if other candidates are removed

Arrow's Impossibility Theorem tells us that each of our 4 voting methods violates at least one of these criteria.

The Electoral System in Canada

In Canadian federal elections, we use the plurality method combined with the **first past the post method**. Voters vote for a candidate in their **riding** (based on their city or geographical location), who represents a certain party. The party with the most first-place votes wins the election and is represented by the prime minister.

The numbers on the map below represent the number of ridings for each province:



How is our Canadian electoral system similar to the warm-up that we did today?

No matter how many votes you get for a certain riding, that riding only has a weight of 1. Therefore, if the second-place party in Toronto had millions of votes, they would not count for anything, but if the first-place party in Yukon had a few thousand votes, they would get a seat (win the election).

Problems

1. Determine the type of ballot used in the following cases:

	Choose one candidate
a)	Bruno Mars The Weeknd Ed Sheeran Kendrick Lamar

b) List your top 3 favourite tennis players in order of preference

	Rank all candidates in order of preference
c)	Famoso Subway The Keg Swiss Chalet

	List all candidates in order of preference
d)	1st
	2nd
	3rd
	4th
	5th

2. Determine the type of outcome from the following elections:

- a) The Oscars choose one winner from a list of nominees
- b) You and your friends rank your top 10 favourite songs
- c) *Dancing with the Stars* ranks the contestants, eliminating one each week.
- d) The top three students are recognized for having the highest marks in math class.

3. Build a preference schedule and then use the plurality method to find out the results of the following election:

Petey's office is trying to decide what type of food to eat at their retreat next month. Their options are pizza (P), hamburgers (H), salad (S), noodles (N) and chicken (C). 49 people choose the following ballot: Pizza, hamburgers, noodles, chicken, salad; 48 people order their choices as hamburgers, salad, chicken, noodles, pizza; and 3 people prefer the order of noodles, hamburgers, salad, chicken, pizza.

4. The Bowerman award is an award given to the top male and female track and field athletes in the American College system (NCAA).

The winner is decided using a preference ballot: voters rank the 3 finalists. Below is the preference schedule for the male finalists in 2017:

Athlete	1st (3 pts)	2nd (2 pts)	3rd (1 pt)	Total points
Christian Coleman	20	3	7	
Fred Kerley	10	20	0	
Lindon Victor	0	7	23	

Based on the Borda count method, who won the Bowerman Men's Award in 2017? Who came 2nd and 3rd?

5. Using the plurality with elimination method, find out who won the rights to host the 2024 Olympic Games (as well as the final rankings):

Number of voters	8	7	5	4	3	2
1st	Paris	LA	Budapest	Hamburg	Budapest	Hamburg
2nd	Rome	Rome	Paris	LA	Hamburg	Paris
3rd	Budapest	Hamburg	LA	Paris	Rome	LA
4th	LA	Budapest	Hamburg	Rome	LA	Budapest
5th	Hamburg	Paris	Rome	Budapest	Paris	Rome

6. The Toronto Maple Leafs are moments away from the 2017 NHL Entry Draft. They are using pairwise comparison to try to figure out in what order the first 5 draft picks will be chosen. The top five draft picks will be chosen from Cale Makar, Nico Hischier, Nolan Patrick, Miro Heiskanen and Elias Pettersson. Here is the preference schedule as voted by TSN analysts:

# of voters	2	6	4	1	1	4	4
1st	Hischier	Patrick	Patrick	Heiskanen	Heiskanen	Makar	Pettersson
2nd	Makar	Hischier	Hischier	Patrick	Makar	Hischier	Heiskanen
3rd	Heiskanen	Heiskanen	Makar	Hischier	Hischier	Pettersson	Makar
4th	Patrick	Makar	Pettersson	Makar	Patrick	Heiskanen	Patrick
5th	Pettersson	Pettersson	Heiskanen	Pettersson	Pettersson	Patrick	Hischier

Who will be drafted in what order?

7. In an election of 7 candidates, how many pairwise comparisons do we need to do to find the winner?
8. In a preference ballot Borda count election of 7 candidates with 32 voters, what is the maximum possible points a candidate can get?
9. How many different ballots are available if there are 4 candidates in an election?
10. For each of the four voting methods, find which of the four fairness criteria are broken.

Solutions

1. a) single-choice ballot
b) truncated preference ballot
c) preference ballot
d) Preference ballot
2. a) winner-only
b) Partial ranking
c) Full ranking
d) Partial ranking

3.

Number of voters	49	48	3
1st	P	H	N
2nd	H	S	H
3rd	N	C	S
4th	C	N	C
5th	S	P	P

Pizza wins with 49 first-place votes under the plurality method. Hamburgers get 48 votes and noodles get 3 votes. Notice though that even though pizza won, it has 51 last-place votes! Looks like the plurality method is an interesting choice.

4.

Athlete	1st (3 pts)	2nd (2 pts)	3rd (1 pt)	Total points
Christian Coleman	20	3	7	73
Fred Kerley	10	20	0	70
Lindon Victor	0	7	23	37

Using the Borda Count Method, Christian Coleman wins the Bowerman award with 73 points. Fred Kerley comes second with 70 points and Lindon Victor comes third with 37 points.

5. Round 1

Candidate	Paris	LA	Budapest	Hamburg	Rome
Number of first-place votes	8	7	8	6	0

Rome has the fewest first-place votes, so they are eliminated. Since they had 0 first-place votes, none are transferred.

Round 2

Candidate	Paris	LA	Budapest	Hamburg
Number of first-place votes	8	7	8	6

Hamburg has the fewest first-place votes (6), so they are eliminated. 4 of their votes go to LA and 2 go to Paris.

Round 3

Candidate	Paris	LA	Budapest
Number of first-place votes	10	11	8

Budapest has the fewest first-place votes (8), so they are eliminated. 5 of their votes go to Paris, and the other 3 go to LA.

Round 4

Candidate	Paris	LA
Number of first-place votes	15	14

Paris has 15 votes, which is more than the majority (there are a total of 29 votes). Therefore, Paris is the host of the 2024 Olympic Games!

	Pairwise comparison	Votes	Winner (points)
6.	Hischier v Patrick	Hischier (7); Patrick (15)	Patrick (1)
	Hischier v Heiskanen	Hischier (16); Heiskanen (6)	Hischier (1)
	Hischier v Makar	Hischier (13); Makar (9)	Hischier (1)
	Hischier v Petersson	Hischier (18); Petersson (4)	Hischier (1)
	Patrick v Heiskanen	Patrick (10); Heiskanen (12)	Heiskanen (1)
	Patrick v Makar	Patrick (11); Makar (11)	Patrick ($\frac{1}{2}$); Makar ($\frac{1}{2}$)
	Patrick v Petersson	Patrick (14); Petersson (8)	Patrick (1)
	Heiskanen v Makar	Heiskanen (12); Makar (10)	Heiskanen (1)
	Heiskanen v Petersson	Heiskanen (10); Petersson (12)	Petersson (1)
	Makar v Petersson	Makar (18); Petersson (4)	Makar (1)

In total, Hischier has 3 points and will be drafted first, Patrick has $2\frac{1}{2}$ points and will be drafted second, Heiskanen has 2 points and will be drafted third, Makar has $1\frac{1}{2}$ points and will be drafted fourth, and Petersson has 1 point and will be drafted fifth.

7. We will need $\frac{7(6)}{2} = 21$ pairwise comparisons to find a winner in an election of 7 candidates.

8. If the winning candidate got first-place votes from all 32 voters, they would have $32 \times 7 = 224$ points.

9. 4 candidates in an election would have a total of $4! = 4 \times 3 \times 2 \times 1 = 24$ different ballots.

10.

Criterion	Plurality	Borda count	Plurality with elimination	Pairwise comparisons
Majority	No	Violated	No	No
Condorcet	Violated	Violated	Violated	No
Monotonicity	No	No	Violated	No
IIA	Violated	Violated	Violated	Violated

Plurality

The plurality method only needs the majority of first place votes, so it does not violate the majority criterion.

The Condorcet candidate will not necessarily win: preferences other than first-place votes are not taken into account.

It does not violate the monotonicity criterion: first-place votes won't be affected by being

moved up (they can't be).

It violates the independence-of-irrelevant-alternatives (IIA) criterion because if we remove a third, irrelevant candidate, those votes could go to the runner-up who could then win.

Borda Count

If many voters vote for a candidate in second or third, they will get a lot of points and can surpass the candidate who wins the majority of first place votes, so the majority criterion is violated.

The Condorcet criterion is also violated: the one who wins in pairwise competition will not necessarily get the most points.

Moving the winner higher up someone's ballot will not influence their win, so the monotonicity criterion is not violated.

The IIA criterion is violated because moving irrelevant alternatives will allocate points differently, which could hurt the winner.

Plurality with Elimination

The majority criterion is not violated since the winner needs to have a majority in order to win the election.

The Condorcet criterion is violated since a candidate could win pairwise against another but then receive more votes from a third candidate who is eliminated, giving them the edge in a future round.

The monotonicity criterion is violated since a third party being eliminated in one round could hurt the winner. This is due to the fact that the voter's second choice is taken into account. In political elections, we call this "splitting the vote".

The IIA criterion is violated since more irrelevant alternatives means more second choices in other places, which could hurt the winner.

Pairwise Comparison

The majority criterion is not broken since first-place votes will help in head-to-head competition: the winner will be higher up on the ballot.

The Condorcet criterion checks pairwise comparison, so it is clearly not violated.

The monotonicity criterion is not violated: once a candidate is higher up on the ballot than another, it doesn't matter how much higher up they are.

The IIA criterion is violated since removing irrelevant alternatives may give other candidates higher votes, which would change how pairwise comparisons work.