Intermediate Math Circles
Fall 2018
Patterns & Counting

Michael Miniou

The Centre for Education in Mathematics and Computing
Faculty of Mathematics
University of Waterloo

December 5, 2018

What Happened Last Week

- Misleading patterns
- Pascal's Triangle
- Triangular Numbers
- Derived formulas for sum of natural numbers and sum of squares
- Review of counting techniques

What Will Happened Today

- Finish review of counting techniques
- Derive the formula for circle chords and region pattern
- Revisit Pascal's Triangle
- Look at a famous pattern

Ready, Set, Dodgeball!

Five people join a dodgeball team. Each person “friends” each of the other people on Facebook. What is the total number of friendships?

On the Road Again

I have five friends I could to invite on a road trip to Detroit. The only problem is my car only has three available seats. How many different car groups are possible?

Tool # 5: Combinations

Remember a combination is an unordered arrangements of objects.

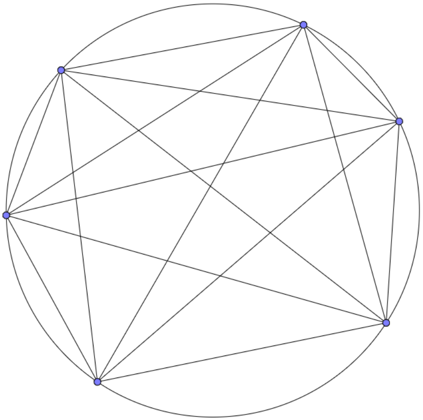
A combination of n different objects taken r at a time without repetitions, that is, the number of r-subsets of a set of n elements is

Notations: ${}_nC_r$ $C(n, r)$ $\binom{n}{r}$

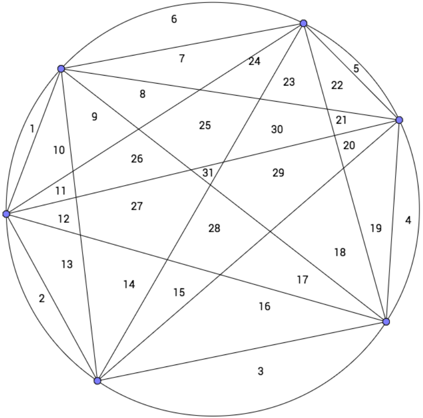
Circle Chords and Regions

Place n distinct points around a circle in such a way that no three chords share a common point in the interior. Draw all the chords. Determine the number of regions all these chords divides the circle into.

The $n = 6$ Case



The $n = 6$ Case


6
7
24
23
22
5
8
9
25
30
21
1
10
26
20
31
29
11
27
12
28
19
4
18
13
17
14
15
2
16
3

Features

Let's focus on three important features of these diagrams:

Features

Let's focus on three important features of these diagrams:

- faces (these are the regions that make up the circle)

Features

Let's focus on three important features of these diagrams:

- faces (these are the regions that make up the circle)
- edges (these are where the "sides" of the faces)

Features

Let's focus on three important features of these diagrams:

- faces (these are the regions that make up the circle)
- edges (these are where the "sides" of the faces)
- vertices (these are the points where the edges meet)

Features

Let's focus on three important features of these diagrams:

- faces (these are the regions that make up the circle)
- edges (these are where the "sides" of the faces)
- vertices (these are the points where the edges meet)

We let F, E, and V be the number of faces, edges, and vertices, respectively.

What happens in the cases that we discussed earlier?

Look Familiar?

Result

In such a drawing with V vertices, E edges, and F faces, then $V - E + F = 1$.

Look Familiar?

Result

In such a drawing with V vertices, E edges, and F faces, then $V - E + F = 1$.

In Ontario's grade 8 curriculum you are expected to

> determine, through investigation using concrete materials, the relationship between the numbers of faces, edges, and vertices of a polyhedron (i.e. *number of faces* + *number of vertices* = *number of edges* + *2*)
>
> (**Sample problem:** Use polyhedrons and/or paper nets to construct the five Platonic solids (i.e. tetrahedron, cube, octahedron, dodecahedron, icosahedron), and compare the sum of the numbers of faces and vertices to the number of edges for each solid.)

Look Familiar?

Euler's Polyhedron Formula

Let F denote the number of faces of a convex polyhedron, E the number of edges, and V the number of vertices.

Then

$$F - E + V = 2.$$

Recall:

- A *polyhedron* is a three-dimensional figure that has polygons as faces.
- Convex has a few definitions saying effectively same thing. For our purposes, it means take any two points inside or on the figure's exterior and connect them with a line, the points on that line will all be in the figure.

Polyhedral Map

How does this relate to our circles and regions?

Polyhedral Map

How does this relate to our circles and regions?

> Any three-dimensional convex polyhedron can be drawn in a plane so that no edges cross. We can call this a *polyhedral map* or *graph*.

Polyhedral Map

How does this relate to our circles and regions?

> Any three-dimensional convex polyhedron can be drawn in a plane so that no edges cross. We can call this a *polyhedral map* or *graph*.

Why does Euler's Formula ($F - E + V = 2$) and our result ($F - E + V = 1$) differ by 1?

Circle Chords and Regions

Now that we have all that out of the way, let's derive the formula for number of regions.

Calculating V in terms of n

Two types of vertices:

Calculating V in terms of n

Two types of vertices:

On circle

Calculating V in terms of n

Two types of vertices:

On circle

Inside circle

Calculating V in terms of n

Two types of vertices:

- On circle
- Inside circle

How many are there?

- On circle:

Calculating V in terms of n

Two types of vertices:

- On circle
- Inside circle

How many are there?

- On circle: n

Calculating V in terms of n

Two types of vertices:

- On circle
- Inside circle

How many are there?

- On circle: n
- Inside circle:

Calculating V in terms of n

Two types of vertices:

On circle

Inside circle

How many are there?

On circle: n

Inside circle: $\binom{n}{4}$

Calculating V in terms of n

Two types of vertices:

On circle

Inside circle

How many are there?

On circle: n

Inside circle: $\binom{n}{4}$

Therefore, $V = n + \binom{n}{4}$

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)
Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)

Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

- On circle:

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)
Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

- On circle: $n+1$

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)
Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

- On circle: $n+1$
- Inside circle:

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)
Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

- On circle: $n + 1$
- Inside circle: 4

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)
Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

- On circle: $n + 1$
- Inside circle: 4

How many edge ends are there in total?

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)

Two types of vertices:

- On circle
- Inside circle

How many edge ends are there at each vertex?

- On circle: $n+1$
- Inside circle: 4

How many edge ends are there in total? $n(n+1)+4\binom{n}{4}$

Calculating E in terms of n

We count "edge ends" and divide by 2. (Why?)
Two types of vertices:

On circle

Inside circle

How many edge ends are there at each vertex?

On circle: $n + 1$

Inside circle: 4

How many edge ends are there in total? $n(n+1) + 4\binom{n}{4}$

Therefore, $E = \frac{1}{2}n(n+1) + 2\binom{n}{4}$

Calculating F in terms of n

Finally,

$$F = 1 + E - V$$

Calculating F in terms of n

Finally,

$$\begin{aligned} F &= 1 + E - V \\ &= 1 + \frac{1}{2}n(n+1) + 2\binom{n}{4} - n - \binom{n}{4} \end{aligned}$$

Calculating F in terms of n

Finally,

$$\begin{aligned} F &= 1 + E - V \\ &= 1 + \frac{1}{2}n(n+1) + 2\binom{n}{4} - n - \binom{n}{4} \\ &= 1 + \frac{1}{2}n(n-1) + \binom{n}{4} \end{aligned}$$

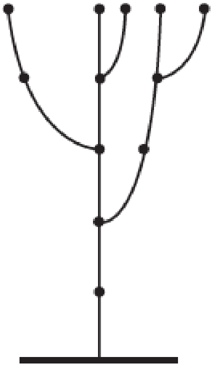
Calculating F in terms of n

Finally,

$$\begin{aligned} F &= 1 + E - V \\ &= 1 + \frac{1}{2}n(n+1) + 2\binom{n}{4} - n - \binom{n}{4} \\ &= 1 + \frac{1}{2}n(n-1) + \binom{n}{4} \\ &= 1 + \binom{n}{2} + \binom{n}{4} \end{aligned}$$

I Speak For The Trees!

A Truffula tree grows according to the following rule. After a branch has been growing for two weeks, it produces a new branch, while the original branch continues to grow every week. The tree has five branches after five weeks, as shown. How many branches, including the main branch, will the tree have at the end of eight weeks?



(A) 19 **(B)** 40 **(C)** 21 **(D)** 13 **(E)** 34

Are We Done Yet?

Answer: (C)

Are We Done Yet?

Answer: (C)

So we think we have found the answer or at least someone has told us we have, but math isn't just about finding the right answer.

Are We Done Yet?

Answer: (C)

So we think we have found the answer or at least someone has told us we have, but math isn't just about finding the right answer.

Fibonacci Numbers

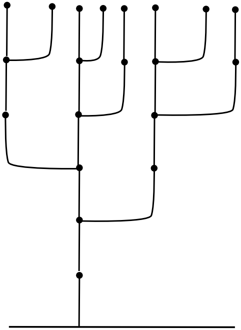
The *Fibonacci numbers*, $F_1, F_2, F_3, \ldots, F_n, \ldots,$ are the sequence of numbers defined by the following recursive equation

$$F_n = F_{n-1} + F_{n-2}$$

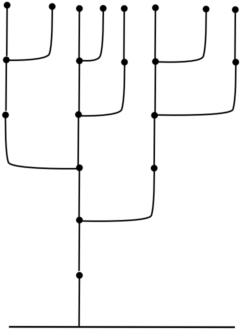
with $F_1 = 1$ and $F_2 = 2$.

The first eight Fibonacci Numbers are $1, 1, 2, 3, 5, 8, 13, 21$.
`http://mathworld.wolfram.com/TriangularNumber.html`

The “Proof”

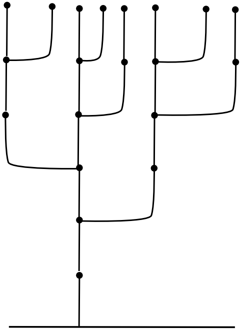


The "Proof"



1

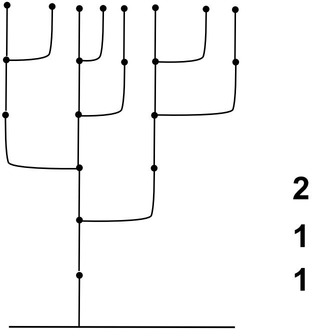
The “Proof”



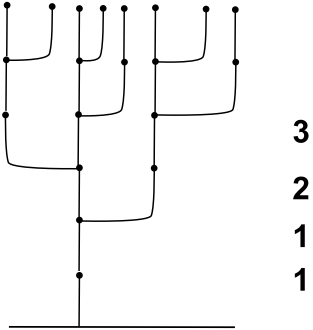
1

1

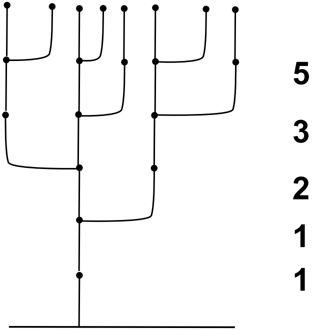
The “Proof”



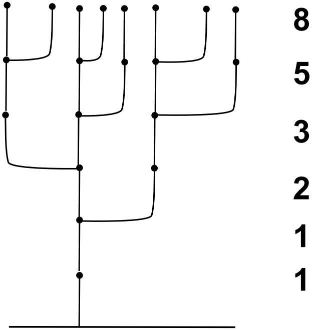
The “Proof”



The "Proof"

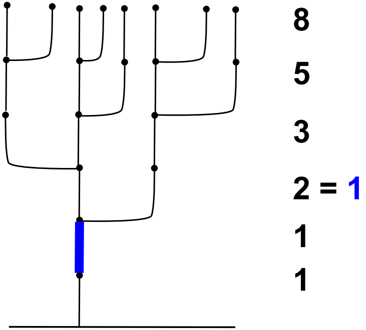


The "Proof"



The “Proof”

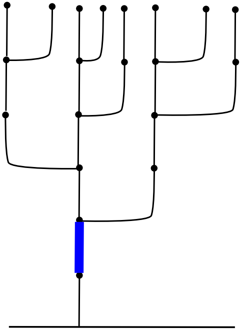
Rule 1A: the original branch continues to grow every week



The “Proof”

Rule 1A: the original branch continues to grow every week

Rule 1B: after a branch has been growing for two weeks, it produces a new branch



8

5

3

2 = 1

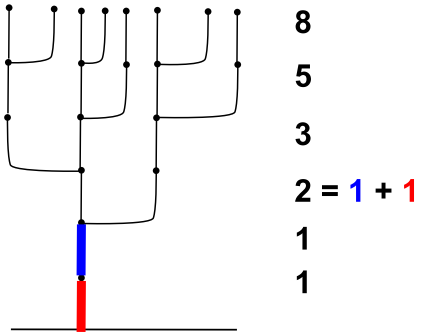
1

1

The "Proof"

Rule 1A: the original branch continues to grow every week

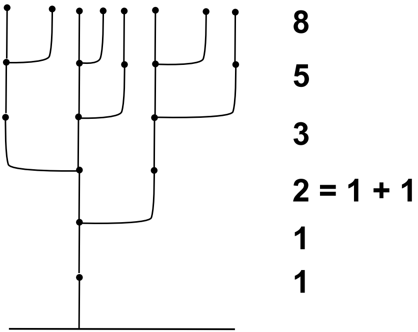
Rule 1B: after a branch has been growing for two weeks, it produces a new branch



The "Proof"

Rule 1A: the original branch continues to grow every week

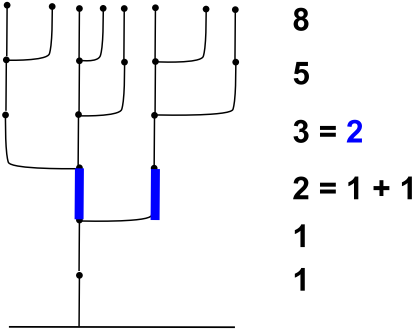
Rule 1B: after a branch has been growing for two weeks, it produces a new branch



The "Proof"

Rule 1A: the original branch continues to grow every week

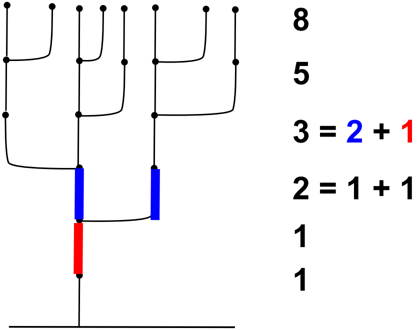
Rule 1B: after a branch has been growing for two weeks, it produces a new branch



The “Proof”

Rule 1A: the original branch continues to grow every week

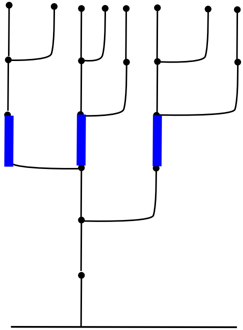
Rule 1B: after a branch has been growing for two weeks, it produces a new branch



The “Proof”

Rule 1A: the original branch continues to grow every week

Rule 1B: after a branch has been growing for two weeks, it produces a new branch



8

5 = 3

3 = 2 + 1

2 = 1 + 1

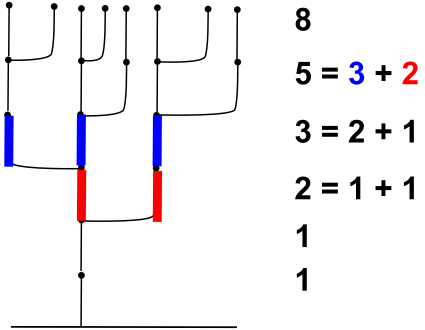
1

1

The "Proof"

Rule 1A: the original branch continues to grow every week

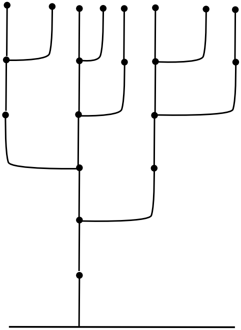
Rule 1B: after a branch has been growing for two weeks, it produces a new branch



The "Proof"

Rule 1A: the original branch continues to grow every week

Rule 1B: after a branch has been growing for two weeks, it produces a new branch



8

5 = 3 + 2

3 = 2 + 1

2 = 1 + 1

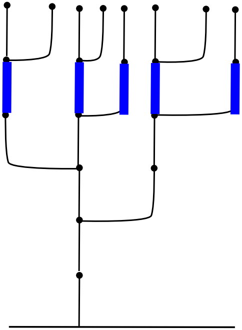
1

1

The "Proof"

Rule 1A: the original branch continues to grow every week

Rule 1B: after a branch has been growing for two weeks, it produces a new branch



8 = 5

5 = 3 + 2

3 = 2 + 1

2 = 1 + 1

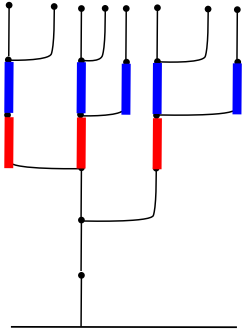
1

1

The “Proof”

Rule 1A: the original branch continues to grow every week

Rule 1B: after a branch has been growing for two weeks, it produces a new branch



$8 = 5 + 3$

$5 = 3 + 2$

$3 = 2 + 1$

$2 = 1 + 1$

1

1

Fibonacci Sequence

End of Week	1	2	3	4	5	6	7	8	9	10	⋯
Branches	1	1	2	3	5	8	13	21	34	55	⋯

Number of branches grows according to the Fibonacci Sequence.

Fibonacci Sequence

End of Week	1	2	3	4	5	6	7	8	9	10	$\cdots$
Branches	1	1	2	3	5	8	13	21	34	55	$\cdots$

Number of branches grows according to the Fibonacci Sequence.

Each number (after the first two) is the sum of the two numbers before it.

Fibonacci Sequence

End of Week	1	2	3	4	5	6	7	8	9	10	$\cdots$
Branches	1	1	2	3	5	8	13	21	34	55	$\cdots$

Number of branches grows according to the Fibonacci Sequence.

Each number (after the first two) is the sum of the two numbers before it.

Mathematicians have known about the sequence for thousands of years. That pre-dates when Fibonacci brought the sequence to the west.

Fibonacci Sequence

End of Week	1	2	3	4	5	6	7	8	9	10	⋯
Branches	1	1	2	3	5	8	13	21	34	55	⋯

Number of branches grows according to the Fibonacci Sequence.

Each number (after the first two) is the sum of the two numbers before it.

Mathematicians have known about the sequence for thousands of years. That pre-dates when Fibonacci brought the sequence to the west.

The sequence and its cousin, the *golden ratio*, are connected and appear in so many seemingly random places.

Patterns in Pascal's Triangle

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1

Patterns in Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

Sum of Squares of Fibonacci Numbers

Show that the sum of the squares of the first n Fibonacci numbers is equal to the product of the n and $n+1$ Fibonacci numbers.

In other words,

$$(F_1)^2 + (F_2)^2 + (F_3)^2 + \cdots + (F_n)^2 = (F_n)(F_{n+1})$$