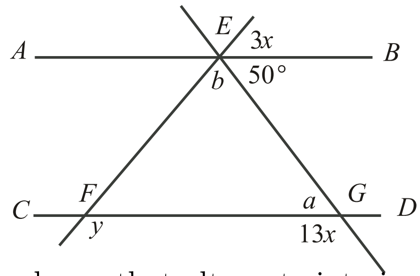
Intermediate Math Circles
Wednesday October 10 2018
Problem Set 1

1. In the diagram, AB is parallel to CD.

   Determine the values of x and y.



Solution

Let $a = \angle EGF$, $b = \angle FEG$. Since $AB \parallel CD$, we know that alternate interior angles are equal so $a = 50°$. Observe that a and the angle $13x$ form a straight angle. Then,

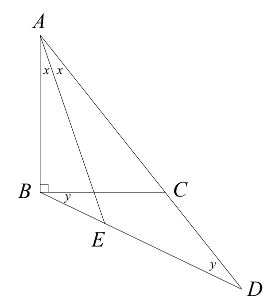
$$\begin{aligned} a + 13x &= 180 \\ 50 + 13x &= 180 \\ x &= 10 \end{aligned}$$

Similarly, using b, the $50°$ angle, and the angle $3x = 30$,

$$\begin{aligned} b + 50 + 30 &= 180 \\ b + 80 &= 180 \\ b &= 100 \end{aligned}$$

Since y is an external angle to $\triangle EFG$, $y = a + b = 150°$.
Therefore, $x = 10°, y = 150°$.

2. Triangle ABC has a right angle at B. AC is extended to D so that $CD = CB$. The bisector of angle A meets BD at E. Prove that $\angle AEB = 45°$.



Solution

Since AE bisects $\angle BAC$, we can let $x = \angle BAE = \angle EAC$.

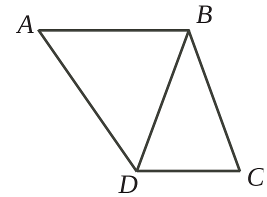
Since CB = CD, $\triangle BCD$ is isosceles so $y = \angle CBD = \angle CDB$.

In $\triangle ABD$, by the sum of interior angles of a triangle,

$$\begin{aligned} 2x + (90 + y) + y &= 180 \\ 90 + 2x + 2y &= 180 \\ 2x + 2y &= 90 \\ x + y &= 45 \end{aligned}$$

In $\triangle ABE$, using the sum of interior angles,

$$\begin{aligned} x + (90 + y) + \angle AEB &= 180 \\ 90 + (x + y) + \angle AEB &= 180 \\ 90 + 45 + \angle AEB &= 180 && \text{(from above)} \\ 45 + \angle AEB &= 90 \\ \angle AEB &= 45° && \text{(as required)} \end{aligned}$$



3. In the diagram, AB is parallel to DC and $AB = BD = BC$. If $\angle A = 52°$, determine the measure of $\angle DBC$.

Solution

$\triangle ABD$ is isosceles since $AB = BD$. Therefore $\angle BDA = \angle BAD = 52°$.

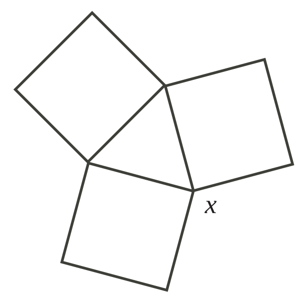
Then in $\triangle BAD$,

$$\begin{aligned}\angle ABD &= 180° - \angle A - \angle BDA \\ &= 180° - 52° - 52° \\ &= 76°\end{aligned}$$

Since AB $\parallel$ DC, we have $\angle BDC = \angle ABD = 76°$.

Since BD = BC, $\triangle BDC$ is isosceles. Therefore, $\angle BDC = \angle BCD = 76°$

Therefore, by sum of interior angles of a triangle, $\angle DBC = 180° - 76° - 76° = 28°$.



4. The diagram shows three squares of the same size. What is the value of x?

Solution

In a square, the corner angles are 90°. The triangle is equilateral (all sides equal), so we know all the angles are equal and hence must be 60° each.

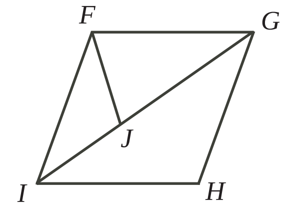
If we look at the place where the triangle and two squares meet (where x is located), we notice it is made up of four angles; two corner angles of a square, one corner angle of a triangle, and x. These four angles form a complete revolution, so they must sum up to 360°.

Then,

$$\begin{aligned}x + 90 + 90 + 60 &= 360 \\ x + 240 &= 360 \\ x &= 120°\end{aligned}$$

Therefore the measure of angle x is 120°.

5. The diagram shows a rhombus $FGHI$ and an isosceles triangle FGJ in which $GF = GJ$. Angle FJI equals $111°$. What is the measure of angle JFI?



Solution

Since $\angle FJI = 111°$ is part of a straight angle with $\angle FJG$, we have that $\angle FJG = 69°$.

We see that because $GF = GJ$, $\triangle FGJ$ is isosceles, with equal base angles $\angle FJG$ and $\angle GFJ$, we get $\angle GFJ = 69°$ and so $\angle FGJ = 42°$

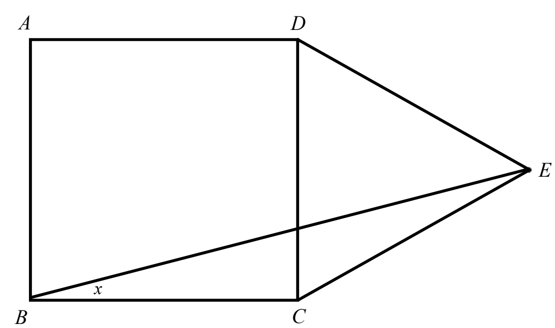
Because $FG \parallel IH$, $\angle FGI = \angle GIH = 42°$. Also, $\triangle IHG$ is isosceles since $GH = HI$, so $\angle IGH = \angle GIH = 42°$

Since $GH \parallel FI$, $\angle FIG = \angle IGH = 42°$.

Using $\triangle FJI$, we see

$$\begin{aligned} \angle FJI + \angle FIJ + \angle JFI &= 180 \\ 111 + 42 + \angle JFI &= 180 \\ \therefore \angle JFI &= 27° \end{aligned}$$

6. $ABCD$ is a square. The point E is outside the square so that CDE is an equilateral triangle. Determine the measure of $\angle BED$.



Solution

Since $ABCD$ is a square, $BC = CD$. Since $\triangle CDE$ is equilateral, $CD = DE = EC$. Therefore, $BC = CD = DE = EC$ and so $BC = EC$.

By the properties of a square, $\angle BCD = 90°$. By the properties of equilateral triangles, $\angle DCE = 60°$. Therefore $\angle BCE = \angle BCD + \angle DCE = 90 + 60 = 150°$.

Since $BC = EC$, $\triangle BCE$ is isosceles. So $\angle EBC = \angle BEC = x$. In this triangle, we have

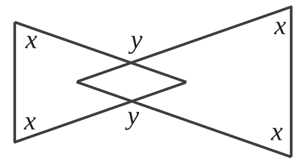
$$\begin{aligned} \angle BCE + x + x &= 180 \\ 150 + 2x &= 180 \\ x &= 15° \end{aligned}$$

So $\angle BEC = x = 15°$.

Note that $60° = \angle DEC = \angle BED + \angle BEC = \angle BED + 15$.

Therefore, $\angle BED = 60 - 15 = 45°$.

7. The diagram shows two isosceles triangles in which the four angles marked x are equal. The two angles marked y are also equal. Find an equation relating x and y.



Solution

Consider the angles opposite to the angles marked y. Since they are opposite angles, they are equal to y.

The quadrilateral formed in the overlap must have angle sum 360°. We know two of the angles are y.

The other two angles are actually the missing angle of the two isosceles triangles. In the left triangle, this angle is $180 - 2x$; for the triangle on the right, it is also $180 - 2x$.

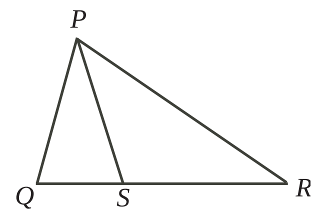
These four angles have to sum to 360°. Therefore,

$$\begin{aligned} y + y + (180 - 2x) + (180 - 2x) &= 360 \\ 2y + 360 - 4x &= 360 \\ 2y &= 4x \\ y &= 2x \end{aligned}$$

$\therefore y = 2x$ is our desired relationship.

8. In the diagram, QSR is a straight line.

$\angle QPS = 12°$ and $PQ = PS = RS$. What is the measure of $\angle QPR$?



Solution

Let $\angle SPR = x$. Then, $\angle QPR = \angle QPS + \angle SPR = 12° + x$.

Since $PS = SR$, $\triangle SPR$ is isosceles and so $\angle PRS = \angle SPR = x$. Since $PS = PQ$, $\triangle PQS$ is isosceles and so $\angle PQS = \angle PSQ = y$.

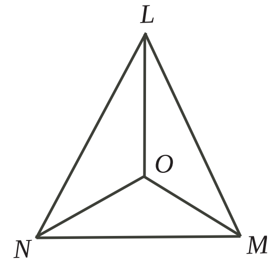
Then

$$\begin{aligned} 12 + y + y &= 180 \\ 2y &= 168 \\ y &= 84° \end{aligned}$$

Since QSR is a straight line, $y = \angle PSQ$ is external to $\triangle PSR$, so $84° = y = x + x = 2x$.

Therefore, $x = 42°$. And $\angle QPR = \angle QPS + \angle SPR = 12 + x = 12 + 42 = 54°$.

9. The three angle bisectors of triangle LMN meet at a point O as shown. Angle LNM is 68°. What is the size of angle LOM?



Solution

Since we are using angle bisectors, let $\angle LNO = \angle ONM = x$, $\angle NLO = \angle OLM = y$, and $\angle LMO = \angle OMN = z$.

But $68° = \angle LNM = \angle NLO + \angle OLM = 2x$, so $x = 34°$.

We also have $\angle LON = 180 - (x + y) = 146 - y$, $\angle LOM = 180 - (y + z)$, and $\angle NOM = 180 - (x + z) = 146 - z$.

$\angle LON$, $\angle NOM$, and $\angle LOM$ form a complete revolution.

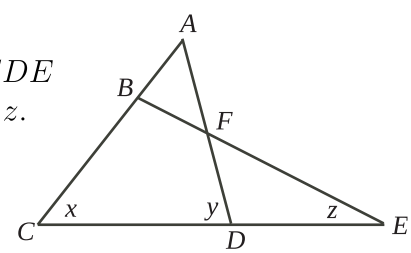
So, $\angle LOM = 360 - \angle LON - \angle NOM = 360 - (146 - y) - (146 - z) = 68 + y + z$

Using the entire triangle,

$$\begin{aligned} \angle LNM + \angle NLM + \angle LMN &= 180 \\ 68 + 2y + 2z &= 180 \\ 2y + 2z &= 112 \\ y + z &= 56 \end{aligned}$$

Therefore, substituting back in, we get $\angle LOM = 68 + 56 = 124°$.

10. In the figure shown, $AB = AF$ and ABC, AFD, BFE, and CDE are all straight lines. Determine an equation relating x, y and z.



Solution

Since $AB = AF$, $\triangle ABF$ is isosceles, so $\angle AFB = \angle ABF = a$.
Since $\angle AFB$ and $\angle DFE$ are opposite angles, $\angle DFE = \angle AFB = a$.

$\angle ABE$ is external to $\triangle CBE$, so $\angle ABE = \angle ACE + \angle BEC$ and $a = x + z$ follows. (1)

$\angle ADC$ is external to $\triangle DFE$, so $\angle ADC = \angle DFE + \angle DEC$ and $y = a + z$ follows. (2)

Substituting (1) into (2) for a, we obtain $y = x + z + z$. Rearranging and simplifying we obtain $x - y + 2z = 0$. This is the equation relating $x,\ y,\ z$.

11. What is the measure of the angle formed by the hands of a clock at 9:10?

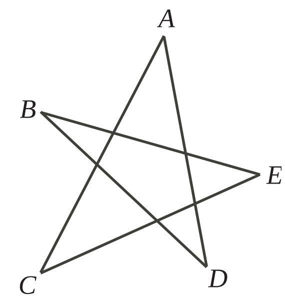
Solution

Every minute after the hour, the minute hand moves $\frac{360}{60} = 6°$ from 12 o'clock. So after 10 minutes, it has moved $10 \times 6 = 60°$ past 12 o'clock.

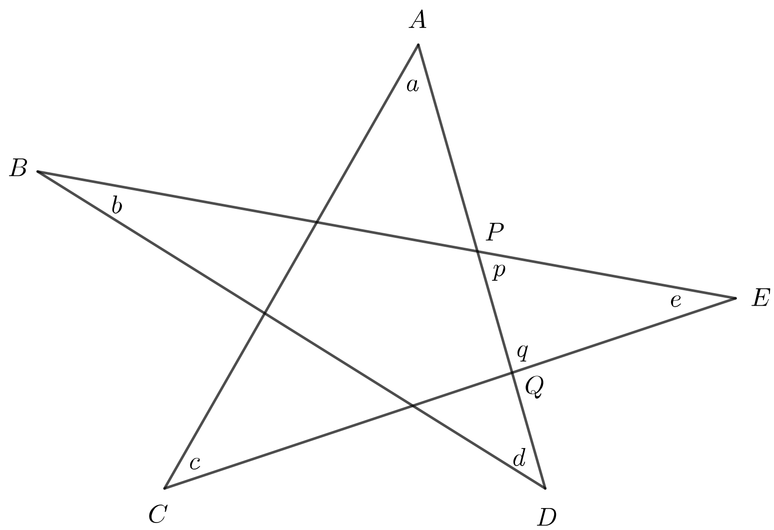
In one hour, the hour hand moves $\frac{360}{12} = 30°$. In ten minutes, it will have moved $\frac{1}{6}$ of this, so it has moved $\frac{1}{6} \times 30 = 5°$ closer to 12 o'clock. 9 o'clock is located $90°$ before 12 o'clock, so the hour hand will be $85°$ before 12 o'clock.

Therefore, the total angle between the hour and minute hand will be $85 + 60 = 145°$.

12. Determine the sum of the angles at A, B, C, D, and E in the five-pointed star shown.



Solution



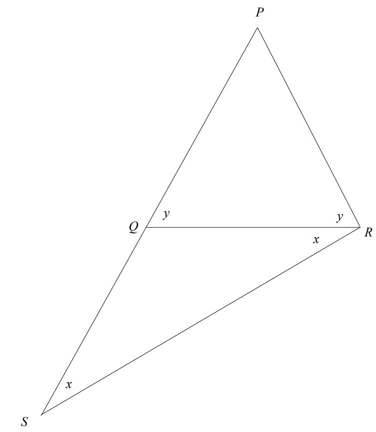
Let $a, b, c, d,$ and e be the angles at A, B, C, D, and E, respectively.
Let P and Q be the points where AD intersects BE and CE, respectively.
Let p represent $\angle EPQ$ and q represent $\angle EQP$.
This information is marked on the above diagram.

$\angle PQE$ is exterior to $\triangle AQC$. Therefore, $\angle PQE = \angle QAC + \angle QCA$. It follows that $q = a + c$. (1)

$\angle QPE$ is exterior to $\triangle PBD$. Therefore, $\angle QPE = \angle PBD + \angle PDB$. It follows that $p = b + d$. (2)

In $\triangle EPQ$, $\angle PEQ + \angle QPE + \angle PQE = 180°$. Substituting, we obtain $e + p + q = 180°$. Further substituting for p from (2) and q from (1), we obtain $e + b + d + a + c = 180°$. That is, the sum of the angles at A, B, C, D, and E is $180°$.

13. In $\triangle PQR$, $PQ = PR$. PQ is extended to S so that $QS = QR$. Prove that $\angle PRS = 3(\angle QSR)$.



Solution

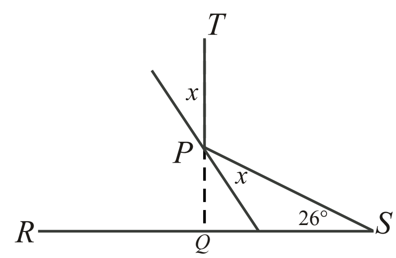
Since $PQ = PR$ and $QS = QR$, we can label the diagram as shown.

Note that $\angle SPR = 180 - 2y$. Using $\triangle SPR$, we see the angle sum gives us

$$\begin{aligned} 180 &= \angle SPR + \angle PSR + \angle PRS \\ 180 &= (180 - 2y) + x + (x + y) \\ 180 &= 180 - y + 2x \\ y &= 2x \end{aligned}$$

So $\angle PRS = x + y = x + 2x = 3x = 3(\angle QSR)$ as required.

14. A beam of light shines from point S, reflects off a reflector at point P, and reaches point T so that PT is perpendicular to RS. What is the value of x?



Solution

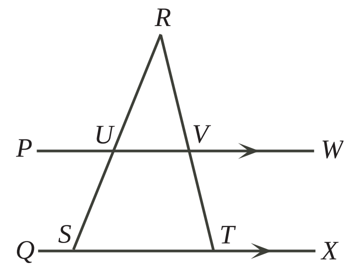
Extend TP to RS, intersecting RS at the point Q as in the diagram. Then $\triangle PQS$ is a right triangle.

Since $\angle TPS$ is exterior to $\triangle PQS$, $\angle TPS = 90 + 26 = 116°$.

Since the reflector forms a straight line, the two angles marked x and $\angle TPS$ form a straight angle. Then

$$\begin{aligned} \angle TPS + x + x &= 180° \\ 116 + 2x &= 180 \\ 2x &= 64 \\ \therefore x &= 32° \end{aligned}$$

15. In the diagram, PW is parallel to QX, S and T lie on QX, and U and V are the points of intersection of PW with SR and TR, respectively. If $\angle SUV = 120°$ and $\angle VTX = 112°$, what is the measure of $\angle URV$?



Solution

Since $PW \parallel QX$, we have

$$\begin{aligned} \angle SUV + \angle TSU &= 180° \\ 120 + \angle TSU &= 180 \\ \angle TSU &= 60° \end{aligned}$$

$\angle RTX$ is exterior to $\triangle RST$. $\therefore \angle RTX = \angle SRT + \angle RST$. (1)

But $\angle RTX = \angle VTX = 112°$ (same angle, given info)

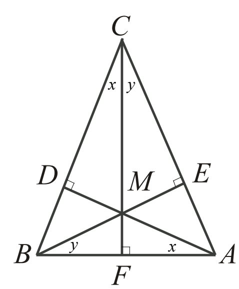
and $\angle RST = \angle TSU = 60°$ (same angle)

$\therefore$ substituting in (1), we have

$$\begin{aligned} \angle SRT + 60 &= 112 \\ \angle SRT &= 52° \end{aligned}$$

But $\angle SRT$ and $\angle URV$ are the same angle. $\therefore \angle URV = 52°$.

16. In the diagram, let M be the point of intersection of the three altitudes of triangle ABC. If $AB = CM$, then what is $\angle BCA$ in degrees?



Solution

Let the three altitudes be AD, BE and CF. In $\triangle CFB$ and $\triangle ADB$, we have $\angle CFB = \angle ADB = 90°$.

Also, $\angle CBF$ and $\angle DBA$ are the same angle, so $\triangle CFB \sim \triangle ADB$.
$\therefore \angle DAB = \angle FCB = x$.

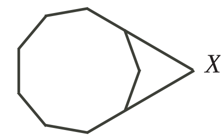
Applying the same argument to $\triangle CFA$ and $\triangle BEA$, we get $\angle FCA = \angle EBA = y$.

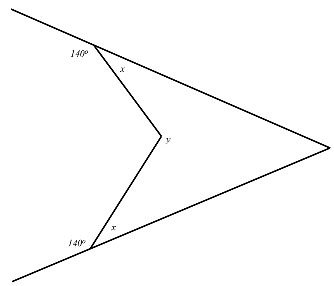
In $\triangle CDM$ and $\triangle ADB$,

$$\begin{aligned} \angle DCM &= \angle DAB = x \\ \angle CDM &= \angle ADB = 90° \\ \therefore \angle CMD &= \angle DBA \\ CM &= BA \\ \therefore \triangle CDM &\cong \triangle ADB \text{ and } CD = DA \end{aligned}$$

So $\triangle CDA$ is right isosceles, hence $\angle DCA = \angle DAC = 45°$. Therefore $\angle BCA = 45°$, since $\angle DCA = \angle BCA$.

17. The diagram shows a regular nonagon with two sides extended to meet at point X. What is the size of the acute angle at X?





Solution

In a regular nonagon (9 sides), the sum of the interior angles is $(9-2) \times 180° = 1260°$. Since the figure is regular, all the interior angles are equal. $\therefore$ each angle is $\frac{1260}{9} = 140°$.

Using our diagram, the two extended sides each form a straight angle. One part of each straight angle is the interior angle, 140°. The other part we will call x must be 40°.

y is part of a revolution; the other part of the revolution is one interior angle of the nonagon, 140°. So $y = 220°$.

The shape containing the angles X, x, y is a quadrilateral. The interior sum must therefore be 360°.

So, $X + x + x + y = 360°$. Plugging in our values for x, y, we see

$$X = 360 - 2x - y = 360 - 80 - 220 = 60°$$

Therefore, $X = 60°$.

18. The angles of a nonagon are nine consecutive numbers. What are these numbers?

Solution

In problem 9, we determined that the sum of the interior angles of a nonagon is 1260°.

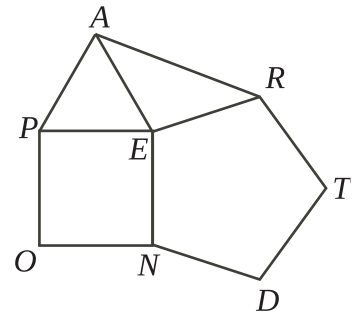
Order the angles from least to greatest, and let the middle angle (the 5th) be x. Since they are consecutive numbers, the angles are

$$\{x-4, x-3, x-2, x-1, x, x+1, x+2, x+3, x+4\}$$

Summing these angles should give us 1260°. If you add the nine angles, you get $9x$. So $9x = 1260$. $\therefore x = 140°$. This is the fifth angle.

Therefore, the list of angles is $\{136°, 137°, 138°, 139°, 140°, 141°, 142°, 143°, 144°\}$.

19. A regular pentagon is a five-sided figure which has all of its angles equal and all of its side lengths equal. In the diagram, $TREND$ is a regular pentagon, PEA is an equilateral triangle, and $OPEN$ is a square. Determine the size of $\angle EAR$.



Solution

Since $\triangle APE$ is equilateral, $\angle PEA = 60°$.

Since $OPEN$ is a square, $\angle PEN = 90°$.

Since $TREND$ is a regular pentagon, with interior angle sum is 540°, each angle equals $540 \div 5 = 108°$. So $\angle NER = 108°$. At E, the angles make a complete rotation, so

$$\begin{aligned}\angle AER &= 360 - \angle PEA - \angle PEN - \angle NER \\ &= 360 - 60 - 90 - 108 \\ &= 102°\end{aligned}$$

Since $\triangle APE$ is equilateral, $AE = PE$. Since $OPEN$ is a square, $PE = EN$. Since $TREND$ is a regular pentagon, $EN = ER$. Therefore AE = PE = EN = ER and $\triangle EAR$ is isosceles. It follows that $\angle EAR = \angle ERA = x$.

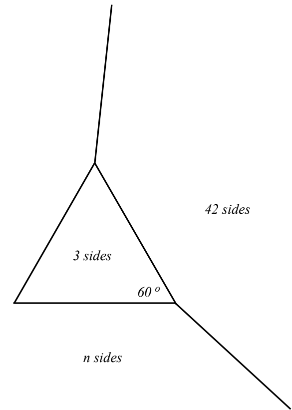
In $\triangle EAR$, we then have

$$\begin{aligned}\angle EAR + \angle ERA + \angle AER &= 180° \\ x + x + 102 &= 180 \\ 2x &= 78 \\ x &= 39\end{aligned}$$

Therefore, $\angle EAR = 39°$

20. Three regular polygons meet at a point and do not overlap. One has 3 sides and one has 42 sides. How many sides does the third polygon have? Can you find other sets of three polygons that have this property?

Solution



Each angle in a regular 3 sided polygon is $\frac{180°}{3} = 60°$.

Each angle in a regular 42 sided polygon is $\frac{180(42-2)}{42} = \frac{1200°}{7}$.

Each angle in a regular n-gon is $\frac{180(n-2)}{n}$.

The 3 angles form a complete revolution.

$$\begin{aligned} \therefore 60° + \frac{1200°}{7} + \frac{180(n-2)}{n} &= 360° \\ \frac{180(n-2)}{n} &= 360° - 60° - \frac{1200°}{7} \\ \frac{180(n-2)}{n} &= \frac{900°}{7} \\ \frac{(n-2)}{n} &= \frac{5°}{7} \\ 7n - 14 &= 5n \\ 2n &= 14 \\ n &= 7 \end{aligned}$$

$\therefore$ it is a 7-sided figure.