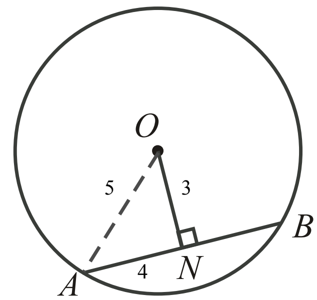
Intermediate Math Circles
Wednesday October 24 2018
Problem Set 3

N.B. Unless otherwise stated, any point labelled O is assumed to represent the centre of the circle.

1. Determine the length of the chord AB if $OA = 5$ and $ON = 3$.



Solution

In the diagram, the radius $OA = 5$ and $ON = 3$.

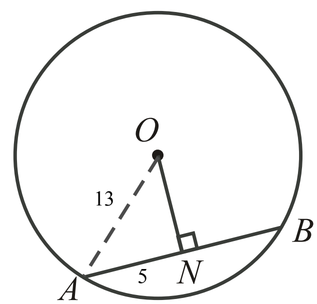
This forms a right triangle $\triangle ONA$. By the Pythagorean Theorem,

$$\begin{aligned} ON^2 + AN^2 &= OA^2 \\ AN^2 &= OA^2 - ON^2 \\ &= 5^2 - 3^2 \\ &= 16 \\ AN &= 4 \qquad (AN > 0) \end{aligned}$$

By the right bisector chord property, N is the midpoint of AB.

So $AN = NB$ and $AB = AN + NB = 4 + 4 = 8$ follows.

2. If $AB = 10$ and $OA = 13$, determine the length of ON.



Solution

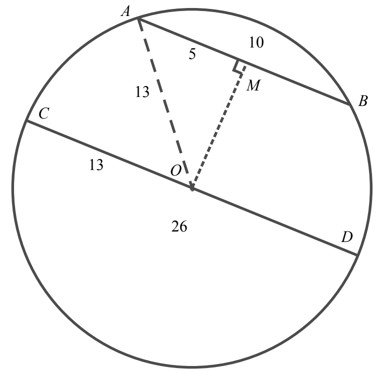
By the right bisector chord property, ON bisects AB.

Hence $AN = NB$. Since $AB = 10$, then $AN = 5$.

Observe $\triangle ONA$ is right angled. By the Pythagorean Theorem,

$$\begin{aligned} ON^2 + AN^2 &= OA^2 \\ ON^2 &= OA^2 - AN^2 \\ ON^2 &= 13^2 - 5^2 \\ &= 144 \\ \therefore ON &= 12 \qquad (ON > 0) \end{aligned}$$

3. A circle has a diameter of length 26. If a chord of the same circle has a length of 10, how far is the chord from the centre?



Solution

Given chord AB, draw diameter $CD \parallel AB$.

Then the distance from AB to O will be the length of the perpendicular bisector of AB (the perpendicular bisector of a chord passes through the centre of a circle).

Let MO be the perpendicular bisector of AB.
M is therefore the midpoint of AB. Then $AM = BM = 5$.

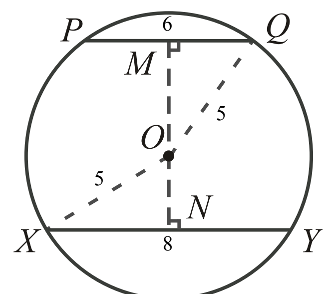
Note OA is a radius of the circle. Since the circle has diameter 26, the radius $OA = 13$.

$MO \perp AB$ by construction, and hence $MO \perp AM$. Then $\triangle AMO$ is right angled. By the Pythagorean Theorem,

$$\begin{aligned} MO^2 + AM^2 &= OA^2 \\ MO^2 + 5^2 &= 13^2 \\ MO^2 &= 144 \\ MO &= 12 \quad (MO > 0) \end{aligned}$$

Therefore the distance from the chord to the centre of the circle is 12 units.

4. Calculate the distance between the parallel chords PQ and XY if $PQ = 6$, $XY = 8$, and the radius of the circle is 5.



Solution

The distance between parallel chords is the length of a perpendicular line segment which extends from one to the other.

Let MN be the perpendicular bisector of PQ. By the right bisector property, this passes through O, and it is also the perpendicular bisector of XY.

Then $PM = MQ = 3$, $NX = NY = 4$ and the length of MN will be the distance between the chords. Observe OX and OQ are radii. Therefore $OX = OQ = 5$.

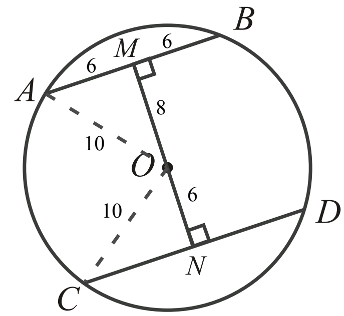
Observe that $\triangle MOQ$ and $\triangle ONX$ are right angled. Applying Pythagorean Theorem to both

$$\begin{aligned} OM^2 + MQ^2 &= OQ^2 \\ OM^2 + 3^2 &= 5^2 \\ OM^2 &= 16 \\ OM &= 4 \quad (OM > 0) \end{aligned}$$

$$\begin{aligned} ON^2 + NX^2 &= OX^2 \\ ON^2 + 4^2 &= 5^2 \\ ON^2 &= 9 \\ ON &= 3 \quad (ON > 0) \end{aligned}$$

Therefore, $MN = MO + ON = 7$, and so the distance between the chords is 7 units.

5. The two parallel chords AB and CD are a distance of 14 apart. If AB has length 12 and the radius of the circle is 10, calculate the length of CD.



Solution

Let MN be the perpendicular bisector of AB, and hence the perpendicular bisector of CD. The length of MN is the distance between the chords, so $MN = 14$.

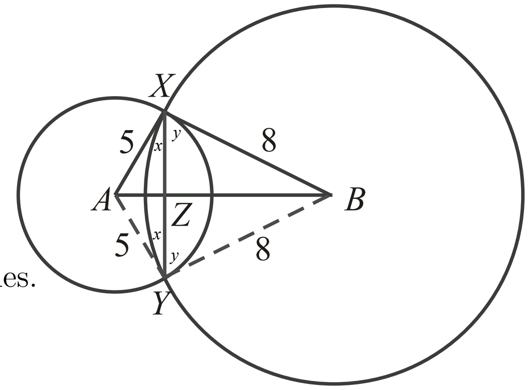
If $AB = 12$, then $AM = BM = 6$. Observe OA and OC are radii. Thus $OA = OC = 10$. Apply the Pythagorean Theorem to $\triangle OMA$.

$$\begin{aligned} OM^2 + AM^2 &= OA^2 \\ OM^2 + 6^2 &= 10^2 \\ OM^2 &= 64 \\ OM &= 8 \quad (OM > 0) \end{aligned}$$

Since $14 = MN = OM + ON = 8 + ON$, then $ON = 6$. Apply the Pythagorean Theorem to $\triangle ONC$

$$\begin{aligned} ON^2 + NC^2 &= OC^2 \\ 6^2 + NC^2 &= 10^2 \\ NC^2 &= 64 \\ NC &= 8 \quad (NC > 0) \end{aligned}$$

So $NC = 8$. But N is the midpoint of CD, so $CN = DN$ and therefore $CD = CN + ND = 16$.



6. Two circles with centre A and B have radii 5 and 8, respectively. The circles intersect at the points X and Y. If $XY = 8$, determine the length of AB, the distance between the centres.

Solution

In the diagram, radii AY and BY are drawn as dotted lines. Let Z mark the intersection of AB and XY.

Since AX and AY are radii of the same circle, $AX = AY$ and hence $\triangle AXY$ is isosceles. So $\angle AXY = \angle AYX = x$. Similarly, since $BX = BY$, $\triangle BXY$ is isosceles and $\angle BXY = \angle BYX = y$.

Observe $\angle AXB = x + y = \angle AYB$. Then by SAS, $\triangle AXB \cong \triangle AYB$.

It follows that, $\angle XAB = \angle YAB$. By ASA, $\triangle AXZ \cong \triangle AYZ$ $(AX = AY, \angle AXZ = \angle AYZ, \angle XAB = \angle YAB)$ and hence $\angle AZX = \angle AZY$.

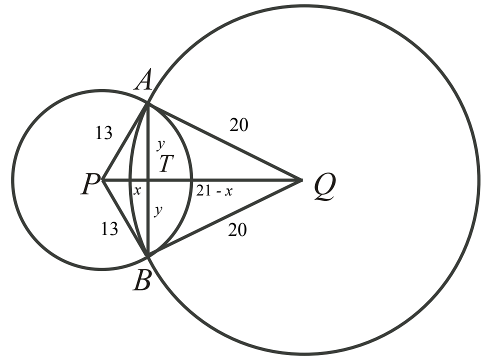
But XY is a straight line, so $\angle AZX = \angle AZY = 90°$, and similarly $\angle BZX = \angle BZY = 90°$.

Thus $\triangle XZA$ and $\triangle XZB$ are right angled. Furthermore, $XZ = ZY = 4$ since $XY = 8$. Applying the Pythagorean Theorem to the two triangles gives

$$\begin{aligned} XZ^2 + AZ^2 &= XA^2 \\ 4^2 + AZ^2 &= 5^2 \\ AZ^2 &= 9 \\ AZ &= 3 \quad (AZ > 0) \end{aligned} \qquad \begin{aligned} XZ^2 + ZB^2 &= XB^2 \\ 4^2 + ZB^2 &= 8^2 \\ ZB^2 &= 48 \\ ZB &= \sqrt{48} \quad (ON > 0) \end{aligned}$$

Therefore the distance between the centres of the circles is $AB = AZ + ZB = 3 + \sqrt{48}$ (Note: $\sqrt{48}$ can be simplified to $3 + 4\sqrt{3}$ so $AB = 3 + 4\sqrt{3}$).

7. In the diagram, $PA = 13$ and $QA = 20$, where P and Q are the centres of the circles. Determine the length of AB if $PQ = 21$.



Solution

Observe that this problem is similar to Problem 6. Thus, using the same reasoning, $AT = TB = y$, $\angle ATP = \angle BTP = \angle ATQ = \angle BTQ = 90°$, and $\triangle ATP$, $\triangle ATQ$ are right angled.

Let $x = PT$. Since $PQ = 21$ and $PT = x$, then $QT = 21 - x$. By the Pythagorean Theorem:

$$PT^2 + AT^2 = AP^2$$
$$x^2 + y^2 = 13^2 \qquad (1)$$

$$QT^2 + AT^2 = AQ^2$$
$$(21 - x)^2 + y^2 = 20^2$$
$$441 - 42x + x^2 + y^2 = 20^2 \qquad (2)$$

From (1), substitute 13^2 for $x^2 + y^2$ in (2) to get

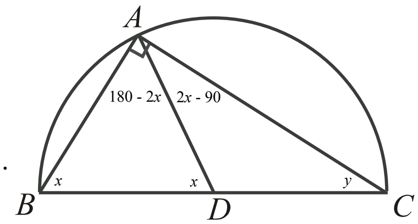
$$\begin{aligned} 441 - 42x + 13^2 &= 20^2 \\ 441 - 42x &= 231 \\ -42x &= 231 - 441 \\ -42x &= -210 \\ x &= 5 \end{aligned}$$

Substitute $x = 5$ into (1) to solve for y

$$\begin{aligned} x^2 + y^2 &= 13^2 \\ y^2 &= 13^2 - 5^2 \\ y^2 &= 144 \\ y &= 12 \qquad (y > 0) \end{aligned}$$

Therefore $AB = AT + TB = 2y = 24$.

8. In the diagram, $\triangle ABC$ is inscribed in the semicircle with centre D. If $AB = AD$, determine the measure of $\angle ACD$.



Solution

Any angle inscribed in a semi-circle is right angled, so $\angle BAC = 90°$.

Since $AB = AD$, $\triangle ABD$ is isosceles and $\angle ABD = \angle ADB = x$.
Then $\angle BAD = 180 - 2x$, and $\angle DAC = 90 - \angle BAD = 2x - 90$.

AD and DC are radii, so $AD = DC$ and thus $\triangle ADC$ is isosceles. Therefore $\angle ACD = \angle CAD$ and $y = 2x - 90$ (1) follows.

$\angle BDA$ is exterior to $\triangle ADC$. It follows that $\angle BDA = \angle DAC + \angle DCA$. Substituting, we obtain $x = 2x - 90 + y = 2x - 90 + 2x - 90 = 4x - 180$. Solving $x = 4x - 180$ we obtain $3x = 180$ and $x = 60$ follows.

Substituting $x = 60$ into (1), $y = 2(60) - 90 = 30$. Therefore, $\angle ACD = 30°$.

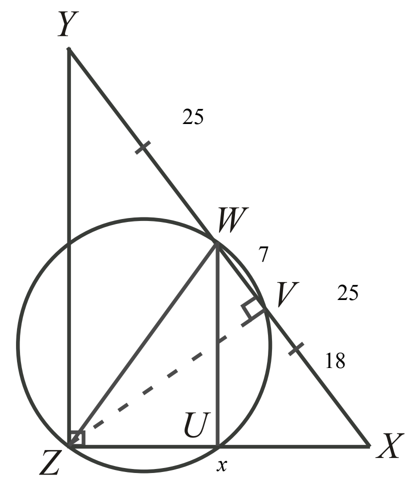
Alternate Solution

Since DA and DB are radii, $AD = DB$. But $AD = DB$. Therefore, $AD = DB = DB$ and $\triangle ABD$ is equilateral. It follows that each angle in $\triangle ABD$ is $60°$.

Since DA and DC are radii, $\triangle ADC$ is isosceles and $\angle CAD = \angle DCA = y$.

Observe that $\angle ADB$ is exterior to $\triangle ADC$. Then $60° = \angle ADB = \angle DAC + \angle DCA = y + y$. So $2y = 60°$ and hence $y = 30°$.

9. In the diagram, $\triangle XYZ$ is right-angled at Z. W is the midpoint of XY, and the circle with diameter ZW intersects WX at V. If $XY = 50$ and $WV = 7$, determine the length of XZ.



Solution

In the diagram, since W is the midpoint of XY, $WY = WX = 25$. Furthermore, if $WV = 7$, then $VX = 18$.

Because V lies on the circle and ZW is a diameter, $\angle WVZ = \angle ZVX = 90^\circ$ (angle inscribed in a semi-circle).

Consider $\triangle WUX$. Since U is on the circumference, $\angle WUZ = \angle WUX = 90^\circ$. Then $\triangle WUX \sim \triangle YZX$ by AAA.

So $\dfrac{WX}{YX} = \dfrac{25}{50} = \dfrac{1}{2} = \dfrac{XZ}{UZ}$. Hence $XZ = 2UZ$ so $UX = UZ$.

But then $\triangle WUX \cong \triangle WUZ$ by SAS since they share common side WU. So $ZW = XW = 25$.

Now, $\triangle WVZ$, $\triangle ZVX$ are right-angled. By the Pythagorean Theorem,

$$\begin{aligned} WV^2 + ZV^2 &= ZW^2 \\ 7^2 + ZV^2 &= 25^2 \\ ZV^2 &= 25^2 - 7^2 \\ ZV^2 &= 576 \end{aligned}$$

and

$$\begin{aligned} ZX^2 &= VX^2 + ZV^2 \\ ZX^2 &= 18^2 + 576 \\ ZX^2 &= 900 \\ \therefore ZX &= 30 \qquad (ZX > 0) \end{aligned}$$

Alternate Solution

In the previous solution, it was shown that $\triangle VZX$ was right angled. But $\triangle VZX$ also shares common angle $\angle YXZ$ with $\triangle ZYX$.

Hence $\triangle VZX \sim \triangle ZYX$. Let $x = ZX$ as shown. Using the properties of similar triangles,

$$\begin{aligned} \frac{VX}{ZX} &= \frac{ZX}{YX} \\ \frac{VX}{x} &= \frac{x}{YX} \\ VX \cdot YX &= x^2 \\ 18 \cdot 50 &= x^2 \\ 900 &= x^2 \\ 30 &= x \qquad (x > 0) \end{aligned}$$

Hence $x = ZX = 30$.