Patterns & Counting Problem Set

Intermediate Math Circles Fall 2018

December 5, 2018

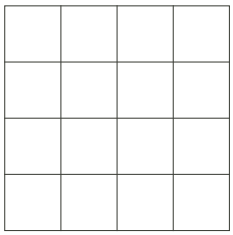
Review of Last Week

1. Evaluate n given $\dfrac{1+2+3+\cdots+(n-1)+n}{3n} = \dfrac{33}{2}$.
2. The symbol $n!$ represents the product of the positive integers from 1 to n. That is, $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. (The symbol $n!$ is read "n factorial".) For example, the value of 4! is 24 because $4 \times 3 \times 2 \times 1 = 24$.
   (a) Determine the value of $\dfrac{7!}{5!}$.
   (b) Determine the positive integer n for which $98! \times 9900 = n!$.
   (c) Determine the positive integer m for which $\dfrac{(m+2)!}{m!} = 40200$.
   (d) Suppose that q is a positive integer and that r is the number for which $(q+2)!-(q+1)! = (q!) \times r$. Show that, for every positive integer q, the number r is an integer which is a perfect square.
3. Starting with 2, the number 2005 can be formed by moving either horizontally, vertically, or diagonally from square to square in grid. How many different paths can be followed to form 2005?

5	5	5	5	5
5	0	0	0	5
5	0	2	0	5
5	0	0	0	5
5	5	5	5	5

4. How rectangles are in the 4 by 4 grid shown?

   Hint: Instead of using cases, try using a technique discussed in the lesson. Remember a rectangle is made up of pairs of parallel lines



5. Scott stacks golfballs to make a pyramid. The first layer, or base, of the pyramid is a square of golfballs and rests on a at table. Each golfball, above the first layer, rests in a pocket formed by four golfballs in the layer below (as shown in Figure 1). Each layer, including the first layer, is completely filled. For example, golfballs can be stacked into a pyramid with 3 levels, as shown in Figure 2. The four triangular faces of the pyramid in Figure 2 include a total of exactly 13 different golfballs. Scott makes a pyramid in which the four triangular faces include a total of exactly 145 different golfballs. How many layers does this pyramid have?

Figure 1

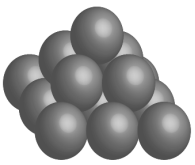


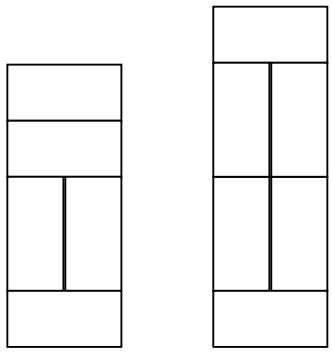
Figure 2

Fibonacci Numbers

1. In the Fibonacci sequence, $1, 1, 2, 3, 5, \ldots$, each term after the second is the sum of the previous two terms. How many of the first 100 terms of the Fibonacci sequence are odd?

2. (a) A window of width 2 and height 5 is made up of five identical 1 by 2 glass panes. One arrangement of the panes is shown below. How many different arrangements of this window are possible?

   (b) A window of width 2 and height 6 is made up of six identical 1 by 2 glass panes. One arrangement of the panes is shown below. How many different arrangements of this window are possible?



2 by 5 2 by 6

   (c) A window of width 2 and height n, where n is a positive integer, is made up of n identical 1 by 2 glass panes. Explain why the number of arrangements is equal to the n^{th} Fibonacci number.

3. The Fibonacci numbers F_n are given by

$$F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n$$

   where $n \geq 1$. Thus the first few terms of the Fibonacci series are $1, 1, 2, 3, 5, 8, 13, \ldots$

   Prove the following identities

   (a) $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

   (b) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

   (c) $F_2 + F_4 + F_6 + \cdots + F_{2n} = F_{2n+1} - 1$

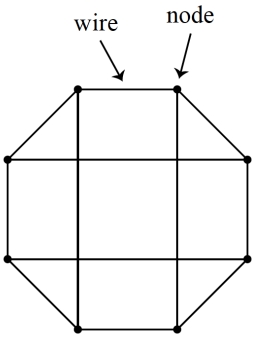
4. A triangle can be formed having side lengths 4, 5 and 8. It is impossible however, to construct a triangle with side lengths 4, 5 and 10. Ron has eight sticks, each having an integer length. He observes that he cannot form a triangle using any three of these sticks as side lengths. What is the shortest possible length of the longest of the eight sticks?

5. In the sequence $1, 2, 3, 1, 4, 3, 7, 4, 11, 7, 18, \cdots, 1364, 3571$, the first two terms are 1 and 2. Every term after the second is formed by combining the previous two terms, alternating between adding and finding the positive difference. Find the sum of this sequence.

Graphs

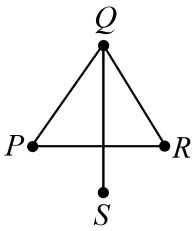
1. A Miniou circuit contains nodes and wires and obeys the following rules:

   - Each wire connects two different nodes.
   - There is at most one wire between each pair of nodes.
   - Exactly three wires are connected to each node.

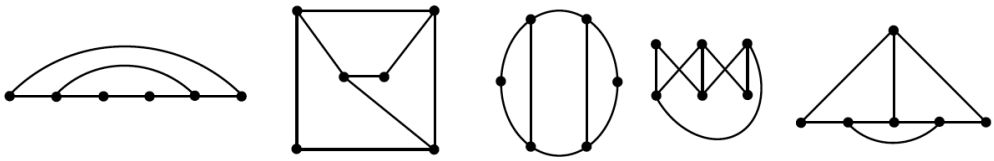
   

   An example of a Miniou circuit is shown. If a Miniou circuit has 13 788 wires, how many nodes does it have?

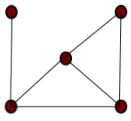
2. Paula, Quinn, Rufus, and Sarah are suspects in a crime. The police found links between exactly four pairs of suspects: Paula and Quinn, Quinn and Rufus, Rufus and Paula, and Quinn and Sarah. These links can be shown in a diagram by drawing a point to represent each suspect and a line or curve joining two points whenever the two corresponding suspects are linked. An example of a drawing that represents this information is:

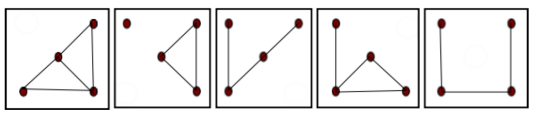
   

   Ali, Bob, Cai, Dee, Eve, and Fay are suspects in a second crime. The police found links between exactly eight pairs of suspects: Ali and Bob, Bob and Cai, Cai and Dee, Dee and Eve, Eve and Fay, Fay and Ali, Ali and Dee, and Bob and Eve. For how many of the following drawings can the six dots be labelled with the names of the six suspects so that each of the eight links given is represented by a line or curve in that drawing?

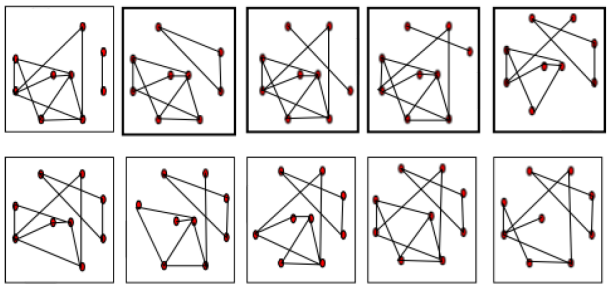
   

3. A graph is a diagram consisting of points (called vertices) and line segments joining some pairs of points (called edges). Starting with an original graph with five vertices, as shown below, five new graphs are formed by removing one of the vertices from the original graph and all of the edges joined to this vertex.

   This process is repeated starting with an original graph with 10 vertices.



How many edges did this second original graph have?

Answers

Review of Last Week

1. $n = 98$
2. (a) 42
   (b) $n = 100$
   (c) $m = 199$
   (d) See solutions
3. 88
4. 100
5. 9

Fibonacci Numbers

1. 67
2. (a) 8
   (b) 13
   (c) See solutions
3. See solutions
4. 21
5. 12 916

Graphs

1. 9192 nodes
2. 2 (the second and fourth drawings)
3. 14