Patterns & Counting Problem Set

Intermediate Math Circles Fall 2018

Triangular Numbers

1. What is the sum of the first 200 positive integers,

$$1 + 2 + 3 + \cdots + 198 + 199 + 200 \ ?$$

2. Calculate the sum of the 50 consecutive integers beginning at 151, that is,

$$151 + 152 + 153 + \cdots + 198 + 199 + 200.$$

3. Starting with the sum of the first 1000 positive integers,
$1 + 2 + 3 + \cdots + 999 + 1000$, every third integer is removed to create the new sum,

$$1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \cdots + 998 + 1000.$$

Calculate the new sum.

4. Determine the value of the expression

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + 10 + 11 - 12 + \cdots + 94 + 95 - 96 + 97 + 98 - 99.$$

The expression consists of 99 terms. The operations alternate between two additions and one subtraction.

5. (a) Prove that the sum of any three consecutive triangular numbers is always one more than three times the middle of these three triangular numbers.

(b) The 3rd, 6th and 8th triangular numbers $(6, 21, 36)$ are said to be in arithmetic sequence because the second minus the first equals the third minus the second, ie. $21-6 = 36-21$. Also, the 8th, 12th and 15th triangular numbers $(36, 78, 120)$ are in arithmetic sequence. Find three other triangular numbers, each larger than 2004, which are in arithmetic sequence.

Patterns in Triangles

1. Consider the following sequence of figures showing arrangements of square tiles:

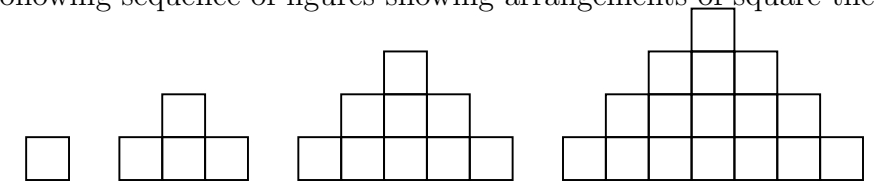


Figure 1 Figure 2 Figure 3 Figure 4

More figures can be drawn, each having one row of tiles more than the previous figure. This new bottom row is constructed using two tiles more than the number of tiles in the bottom row of the previous figure.

(a) Determine the number of tiles in Figure 5.

(b) Determine the number of tiles in the bottom row of Figure 10.

(c) Determine the difference between the total number of tiles in Figure 11 and the total number of tiles in Figure 9.

2. The even positive integers are listed in order and arranged into rows, as shown, and described below.

Row Number				
1	2			
2	4	6		
3	8	10	12	
4	14	16	18	20
		⋮		

Each new row includes one more integer than the previous row. The last number in each row is the product of the row number and the next largest integer. For example, the last number in the 4th row is 4×5. You may use this fact without proving it.

(a) List the numbers in the 7th row of the table.

(b) What are the first and last numbers in the 100th row of the table?

(c) The last number in row r is L. The first number in row $(r + 2)$ is F. Determine the smallest possible value for r such that $F + L$ is at least 2013.

3. A list of integers is written in a table, row after row from left to right. Row 1 has the integer 1. Row 2 has the integers 1, 2 and 3. Row n has the consecutive integers beginning at 1 and ending at the n^{th} odd integer. In the table, the 9^{th} integer to be written is 5, and it appears at the end of Row 3. In general, after having completed n rows, a total of n^2 integers have been written.

Row 1	1						
Row 2	1	2	3				
Row 3	1	2	3	4	5		
Row 4	1	2	3	4	5	6	7
⋮							

(a) What is the 25^{th} integer written in the table and in which row does the 25^{th} integer appear?

(b) What is the 100^{th} integer written in the table?

(c) What is the 2017^{th} integer written in the table?

(d) In how many of the first 200 rows does the integer 96 appear?

Counting

1. The digits 1,2,3,4 can be arranged to form twenty-four different four digit numbers. If these numbers are then listed from smallest to largest, in what position is 3142?

2. Solve each of the following for n

(a) $(n-2)! = 5040$ (b) $\dfrac{(n+3)!}{(n+2)!} = 8$ (c) $\dfrac{(n+1)!}{(n-1)!} - \dfrac{n!}{(n-1)!} = 289$

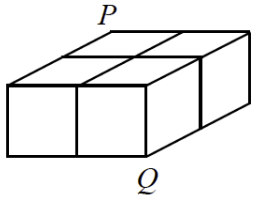
3. Let n be a natural number. As n becomes really large which expression in terms of n is larger and why?

   (a) $n!$ or 2^n (b) $n!$ or 3^n (c) $n!$ or n^n

4. A hardware store sells single digits to be used for house numbers. There are five 5s, four 4s, three 3s, and two 2s available. From this selection of digits, as customer is able to purchase his three-digit house number. Determine the number of possible house numbers this customer could form.

5. Five friends are going to see a movie. They sit in a row of six seats. Assuming they take up one seat each, how many different seating arrangements are possible?

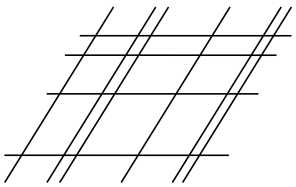
6. Grid lines drawn on three faces of a rectangular prism, as shown. A squirrel walks from P to Q along the edges and grid lines in such a way that she is always getting closer to Q and farther away from P. How many different paths from P to Q can the squirrel take?



7. The word EUCLID can be spelled by tracing paths through the given array of letters. Steps to adjacent letters horizontally, vertically, or diagonally are allowed. Determine the number of different paths which spell the word EUCLID.

E	E	E	D	D	D
E	U	U	I	I	D
E	U	C	L	I	D
E	U	C	L	I	D
E	U	U	I	I	D
E	E	E	D	D	D

8. How many parallelograms of any size are formed by the intersecting parallel lines shown in the figure shown to the right?



9. Ten points are marked on a circle

   (a) How many line segments can be formed using any two of the ten points?

   (b) How many triangles can be formed using these three points?

   (c) Suppose one of the points is labelled as A. How many triangles contain point A as a vertex?