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Mathematics and Computing

Grade 6 Math Circles

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Area of Triangles

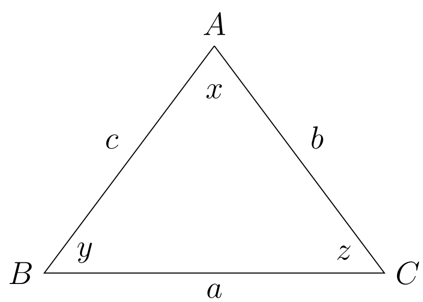
Today we will learn how to find the area of a triangle in several different ways. Before we begin calculating areas, let's review some knowledge about triangles.

The Basics

A **triangle**, denoted $\triangle ABC$, has the following properties:

- $\triangle ABC$ has 3 vertices, 3 sides, and 3 interior angles
- The sum of the interior angles of $\triangle ABC$ is 180°

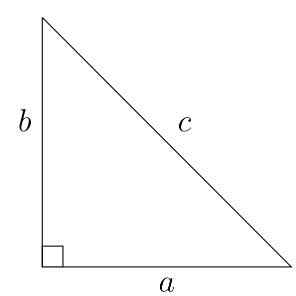
We can label a triangle, $\triangle ABC$, as follows:



$$\text{Sum of interior angles} = 180° = \angle BAC + \angle ABC + \angle BCA = x + y + z$$

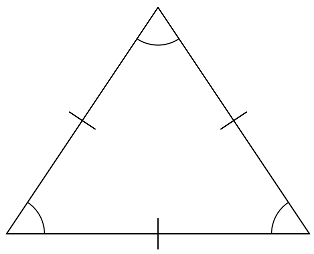
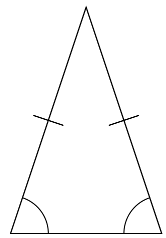
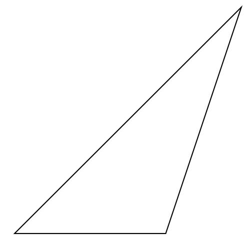
Pythagorean Theorem

Pythagorean Theorem is a very popular mathematical theorem about the three sides of a right triangle. It states that the square of the hypotenuse is equal to the sum of the squares of the other two sides. If we write this using mathematical notation, we get:



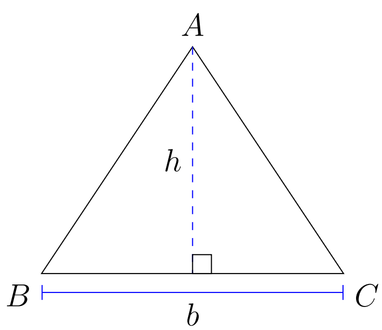
$$a^2 + b^2 = c^2$$

3 Types of Triangles

Equilateral	Isosceles	Scalene
		
All side lengths are equal All angles are equal	Two side lengths are equal Two angles are equal	All side lengths are different All angles are different

Find the Area of a Triangle

Method 1: Basic Formula

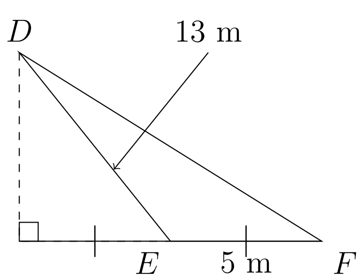


Basic Formula for Area of a Triangle

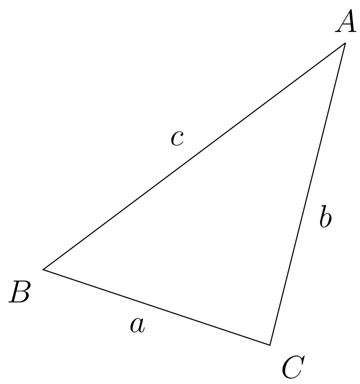
Let b be the base of the triangle and h be the height of the triangle. The area of the triangle, $A_{\triangle ABC}$, is...

$$A_{\triangle ABC} = \frac{b \times h}{2} = \frac{1}{2}(b \times h)$$

Example 1: Find the area of $\triangle DEF$.



Method 2: Heron's Formula



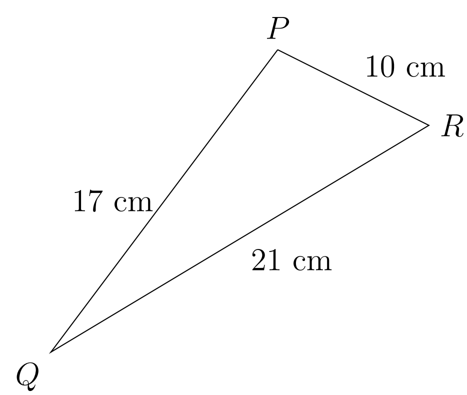
Heron's Formula

Let a, b, and c be the side lengths of $\triangle ABC$. Then,

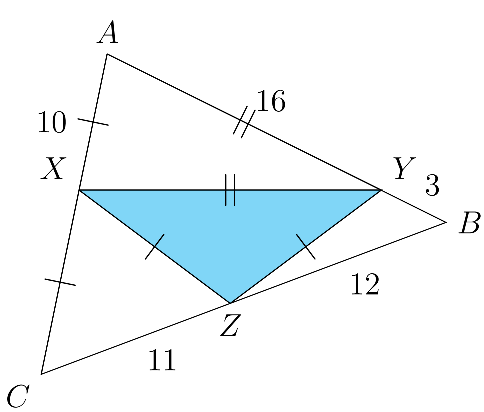
$$A_{\triangle ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the *semi-perimeter* of $\triangle ABC$.

Example 2: Find the area of $\triangle PQR$:



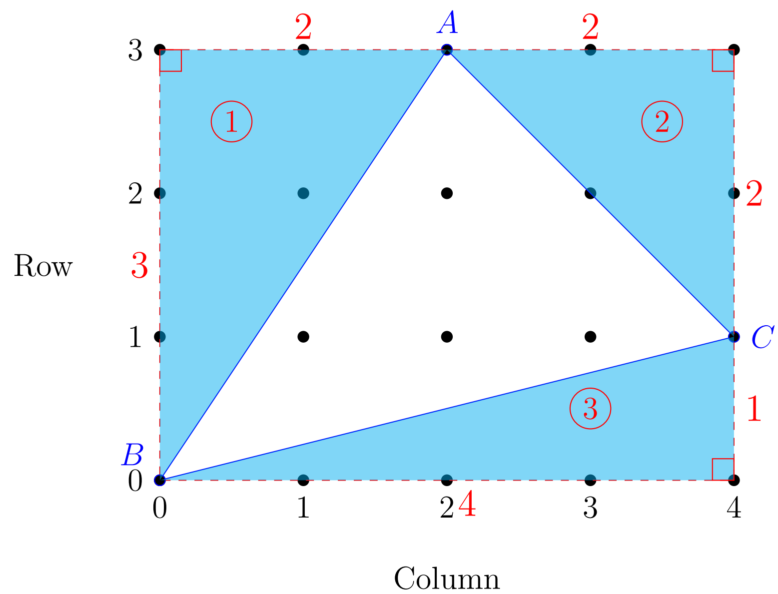
Example 3: Suppose you have $\triangle ABC$. What is the area of $\triangle XYZ$? (Image is not to scale.)



Method 3: Complete the Rectangle

Suppose we have the coordinates of $\triangle ABC$. We do not know what the side lengths are but we are given the coordinates, kind of like the game *Battleship*! Draw the triangle on the graph below.

$A(2{,}3)$ $\qquad$ $B(0{,}0)$ $\qquad$ $C(4{,}1)$



Let's find the area of $\triangle ABC$. Since we do not know the side lengths, we will enclose $\triangle ABC$ inside a rectangle. Notice the rectangle is made up of four triangles: $\triangle ABC$, ①, ②, and ③. How can we calculate the area of $\triangle ABC$?

Subtract the areas of ①, ②, and ③ from the area of the rectangle!

$$\begin{aligned}
A_{rectangle} &= \text{length} \times \text{width} = 4 \times 3 = 12 \\
A_{①} &= \frac{b \times h}{2} = \frac{2 \times 3}{2} = 3 \\
A_{②} &= \frac{2 \times 2}{2} = 2 \\
A_{③} &= \frac{4 \times 1}{2} = 2 \\
A_{\triangle ABC} &= A_{rectangle} - A_{①} - A_{②} - A_{③} \\
A_{\triangle ABC} &= 12 - 3 - 2 - 2 \\
A_{\triangle ABC} &= 5 \text{ units}^2
\end{aligned}$$

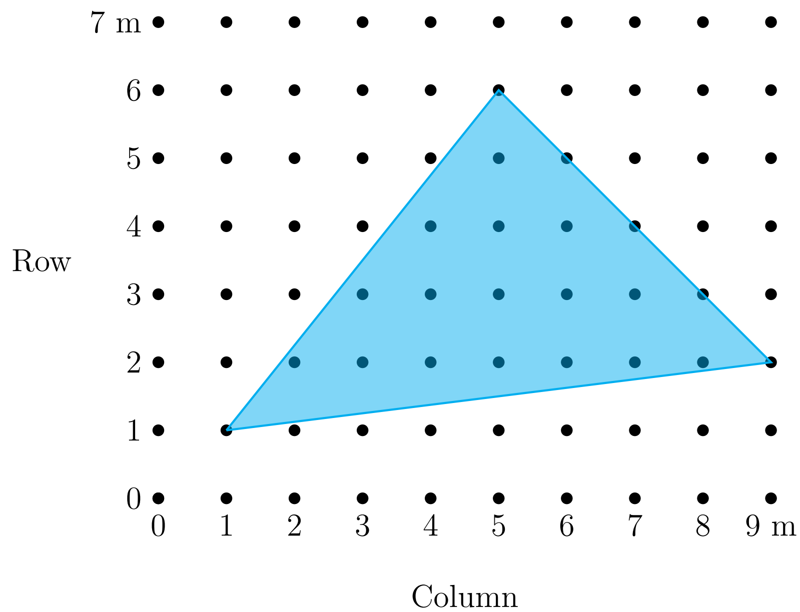
Complete the Rectangle

If we have a triangle, $\triangle ABC$, and we enclose it in a rectangle, then

$$\mathrm{A}_{\triangle ABC} = \mathrm{A}_{rectangle} - \mathrm{A}_{\textcircled{1}} - \mathrm{A}_{\textcircled{2}} - \mathrm{A}_{\textcircled{3}}$$

where $\textcircled{1}$, $\textcircled{2}$, and $\textcircled{3}$ are triangles that form a rectangle with $\triangle ABC$.

Example 4: Alain has a 7 m $\times$ 9 m backyard and decides to construct a triangular-shaped patio. He plots his backyard plan on a graph as follows:



What is the area of his patio? How much space does he have left in his backyard?

Method 4: Shoelace Theorem

Also known as "Shoelace Formula," or "Gauss' Area Formula"

Shoelace Theorem (for a Triangle)

Suppose a triangle has the following coordinates: (a_1, b_1), (a_2, b_2), (a_3, b_3) where $a_1, a_2, a_3, b_1, b_2,$ and b_3 can be any positive number. Then,

$$\text{A}_3 = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}\left|(a_2b_1 + a_3b_2 + a_1b_3) - (a_1b_2 + a_2b_3 + a_3b_1)\right|$$

where $|a|$ is called the *absolute value* of a. (i.e. $|-3| = 3$, $|4| = 4$)

Example 5: Given the coordinates below, what is the area of $\triangle ABC$?

A(3,4) B(7,9) C(11,4)

We can actually generalize the Shoelace Theorem and use it to find the area of a shape with any number of sides!

Shoelace Theorem (for a shape with n sides)

Suppose a shape has n sides, so there are n sets of coordinates: $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$. Then,

$$\mathrm{A}_n = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}\left|(a_2b_1 + a_3b_2 + \ldots + a_1b_n) - (a_1b_2 + a_2b_3 + \ldots + a_nb_1)\right|$$

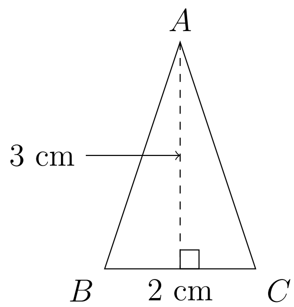
Example 6: Calculate the area of a hexagon with the following coordinates:

(3,9), (9,9), (11,5), (9,1), (3,1), (1,5)

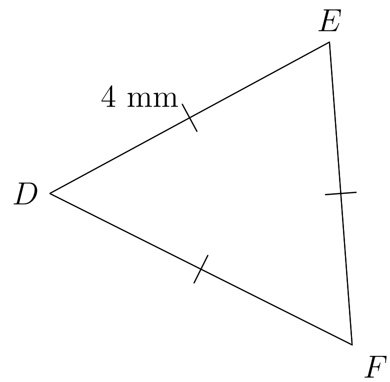
Problem Set

1. Find the area of the following triangles:

(a)

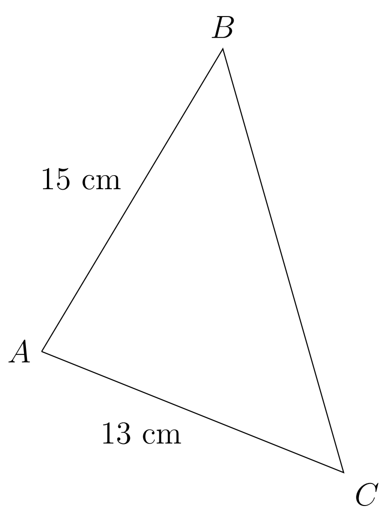


(b)



(c) $\triangle DEF$ where $D(2,8), E(8,0), F(4,4)$

2. Given the perimeter, $P_{\triangle ABC} = 42$ cm, what is the area of $\triangle ABC$?



3. Given $DE = 12$, $EF = 16$, and $FD = 20$, what is the area of $\triangle DEF$?

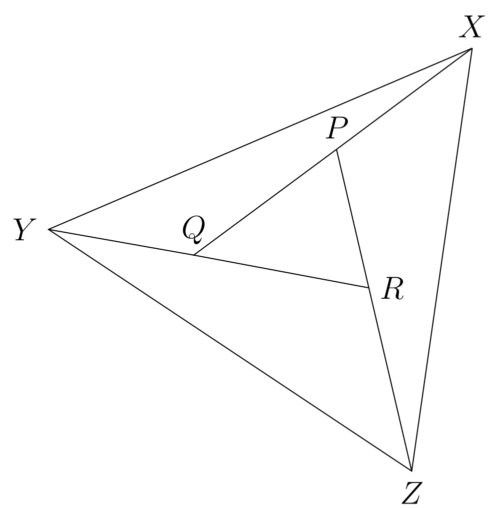
4. The perimeter of $\triangle STU$ is 32 cm. If $\angle STU = \angle SUV$ and $TU = 12$ cm, what is the area of $\triangle STU$?

5. Find the area of a triangle with the following coordinates:

(2,3), (6,3), (8,7)

6. Locations on the main floor of a house are described using coordinate geometry. Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom's base is located at (1,1). Vroom starts at its base and moves to the following points, in order, before returning to its base: (24,10), (20,15), (12,16). What is the area of the figure that Vroom has traced out?

7. $\triangle PQR$ has side QP extended to X so that $QP = PX$, PR extended to Z so that $PR = RZ$, and RQ extended to Y so that $RQ = QY$. If the area of $\triangle XYZ$ is 420 units^2, calculate the area of $\triangle PQR$. Image is not to scale. (*Problems, Problems, Problems, Volume 5: page 30, question 12*)



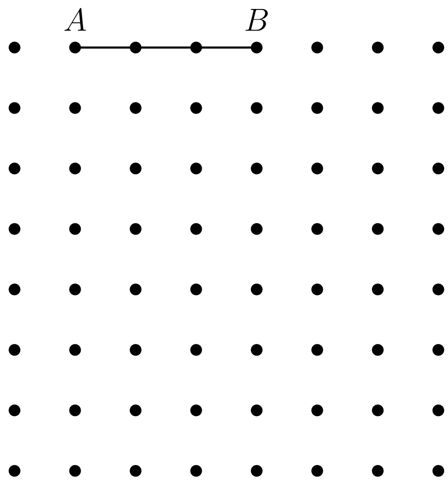
8. * Bernadine and Fred are playing *Battleship: Geometry Edition* where ships sink with one hit. Fred has a triangle-shaped ship with the following coordinates:

$$A(1,7),\ B(4,7),\ C(1,x)$$

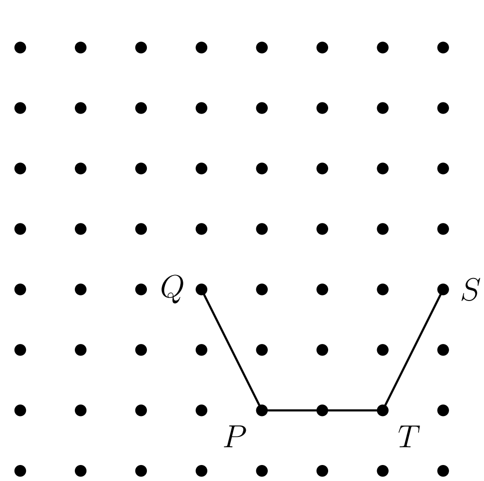
and Bernadine has a pentagon-shaped ship with the following coordinates:

$$P(4,1),\ Q(3,3),\ R(5,y),\ S(7,3),\ T(6,1)$$

Fred's Battle Grid



Bernadine's Battle Grid



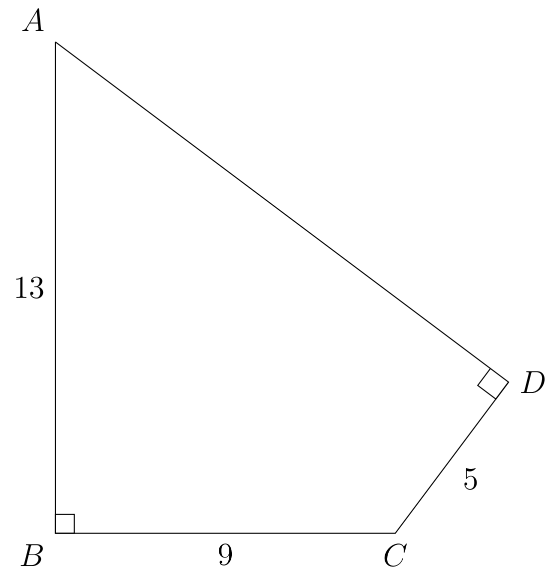
(a) Fred's ship is in the shape of a right-angled triangle. If the area of Fred's ship is 6 units^2, what is the value of x? Draw the rest of Fred's ship on his grid.

(b) Bernadine's ship is in the shape of a pentagon. If the area of Bernadine's ship is 8 units^2, what is the value of y? Draw the rest of Bernadine's ship on her grid.

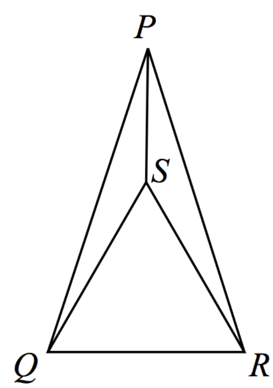
(c) If Fred attacked Bernadine at (4,3), does Fred sink Bernadine's ship?

(d) If Bernadine attacked Fred at (2,6), does Bernadine sink Fred's ship?

9. * In the diagram, what is the area of quadrilateral $ABCD$?



10. ** In the diagram, $\triangle PQR$ is isosceles with $PQ = PR = 39$ and $\triangle SQR$ is equilateral with side length 30. What is the area of $\triangle PQS$? (Round your answer to the nearest whole number.)



11. *** A star is made by overlapping two identical equilateral triangles, as shown below. The entire star has an area of 36 units^2. What is the area of the shaded region?