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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

Fall 2018 - November 13/14

Area of Triangles

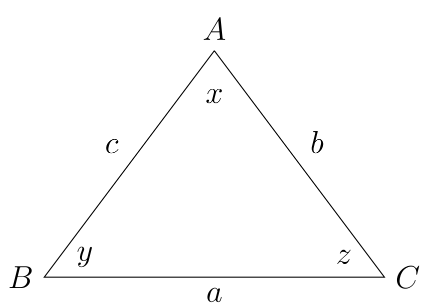
Today we will learn how to find the area of a triangle in several different ways. Before we begin calculating areas, let's review some knowledge about triangles.

The Basics

A **triangle**, denoted $\triangle ABC$, has the following properties:

- $\triangle ABC$ has 3 vertices, 3 sides, and 3 interior angles
- The sum of the interior angles of $\triangle ABC$ is 180°

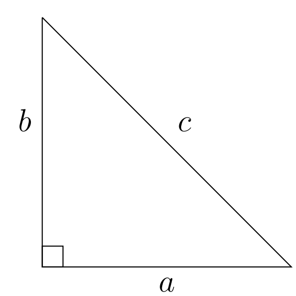
We can label a triangle, $\triangle ABC$, as follows:



$$\text{Sum of interior angles} = 180° = \angle BAC + \angle ABC + \angle BCA = x + y + z$$

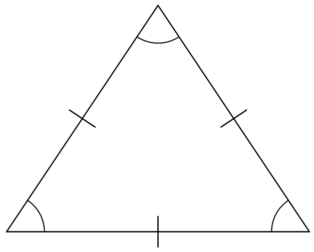
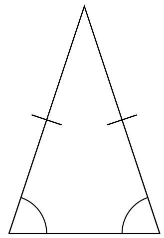
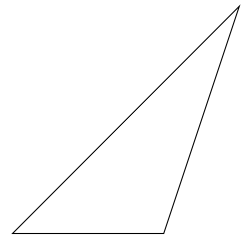
Pythagorean Theorem

Pythagorean Theorem is a very popular mathematical theorem about the three sides of a right triangle. It states that the square of the hypotenuse is equal to the sum of the squares of the other two sides. If we write this using mathematical notation, we get:



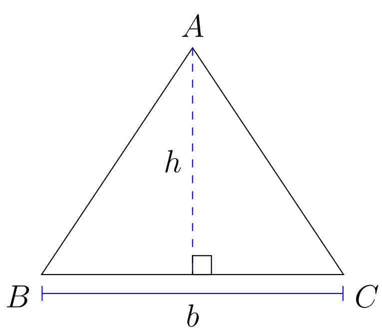
$$a^2 + b^2 = c^2$$

3 Types of Triangles

Equilateral	Isosceles	Scalene
		
All side lengths are equal All angles are equal	Two side lengths are equal Two angles are equal	All side lengths are different All angles are different

Find the Area of a Triangle

Method 1: Basic Formula

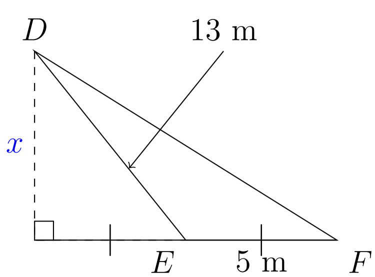


Basic Formula for Area of a Triangle

Let b be the base of the triangle and h be the height of the triangle. The area of the triangle, $A_{\triangle ABC}$, is...

$$A_{\triangle ABC} = \frac{b \times h}{2} = \frac{1}{2}(b \times h)$$

Example 1: Find the area of $\triangle DEF$.

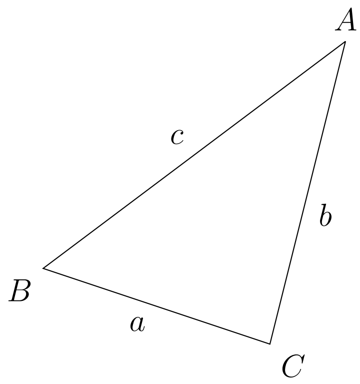


Using Pythagorean Theorem,

$$\begin{aligned} 5^2 + x^2 &= 13^2 \\ 25 + x^2 &= 169 \\ x^2 &= 144 \\ x &= 12 \text{ m} \end{aligned}$$

So, $A_{\triangle DEF} = \frac{5 \times 12}{2} = 30 \text{ m}^2$

Method 2: Heron's Formula



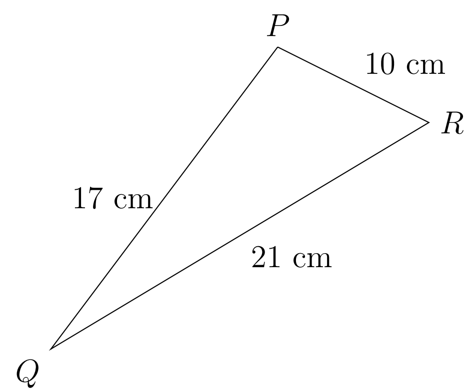
Heron's Formula

Let a, b, and c be the side lengths of $\triangle ABC$. Then,

$$A_{\triangle ABC} = \sqrt{s(s-a)(s-b)(s-c)}$$

where $s = \frac{1}{2}(a+b+c)$ is the *semi-perimeter* of $\triangle ABC$.

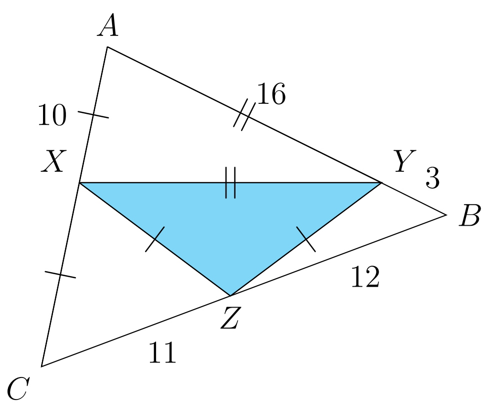
Example 2: Find the area of $\triangle PQR$:



$s = \frac{1}{2}(10+21+17) = 24$

$$\begin{aligned} \text{A}_{\triangle PQR} &= \sqrt{24(24-10)(24-21)(24-17)} \\ &= \sqrt{24(14)(3)(7)} \\ &= \sqrt{7056} \\ \text{A}_{\triangle PQR} &= 84 \text{ cm}^2 \end{aligned}$$

Example 3: Suppose you have $\triangle ABC$. What is the area of $\triangle XYZ$? (Image is not to scale.)



Notice that $\triangle AXY$ is isosceles $\Rightarrow XY = 16$. Then, we see that $AX = XC = ZX = YZ = 10$. Now we can use Heron's formula as follows:

$$s = \frac{1}{2}(XY + YZ + ZX) = \frac{1}{2}(16+10+10) = 18$$

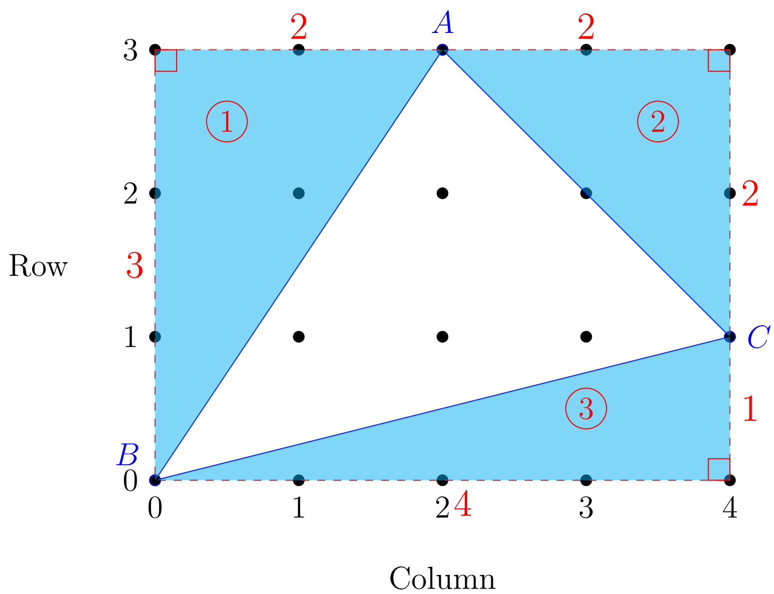
$$\begin{aligned} \text{A}_{\triangle XYZ} &= \sqrt{18(18-16)(18-10)(18-10)} \\ &= \sqrt{18(2)(8)(8)} \\ &= \sqrt{2304} \\ &= 48 \text{ units}^2 \end{aligned}$$

The area of $\triangle XYZ$ is 48 units^2.

Method 3: Complete the Rectangle

Suppose we have the coordinates of $\triangle ABC$. We do not know what the side lengths are but we are given the coordinates, kind of like the game *Battleship*! Draw the triangle on the graph below.

$A(2,3)$ $B(0,0)$ $C(4,1)$



Let's find the area of $\triangle ABC$. Since we do not know the side lengths, we will enclose $\triangle ABC$ inside a rectangle. Notice the rectangle is made up of four triangles: $\triangle ABC$, ①, ②, and ③. How can we calculate the area of $\triangle ABC$?

Subtract the areas of ①, ②, and ③ from the area of the rectangle!

$$
\begin{aligned}
A_{rectangle} &= \text{length} \times \text{width} = 4 \times 3 = 12 \\
A_{①} &= \frac{b \times h}{2} = \frac{2 \times 3}{2} = 3 \\
A_{②} &= \frac{2 \times 2}{2} = 2 \\
A_{③} &= \frac{4 \times 1}{2} = 2 \\
A_{\triangle ABC} &= A_{rectangle} - A_{①} - A_{②} - A_{③} \\
A_{\triangle ABC} &= 12 - 3 - 2 - 2 \\
A_{\triangle ABC} &= 5 \text{ units}^2
\end{aligned}
$$

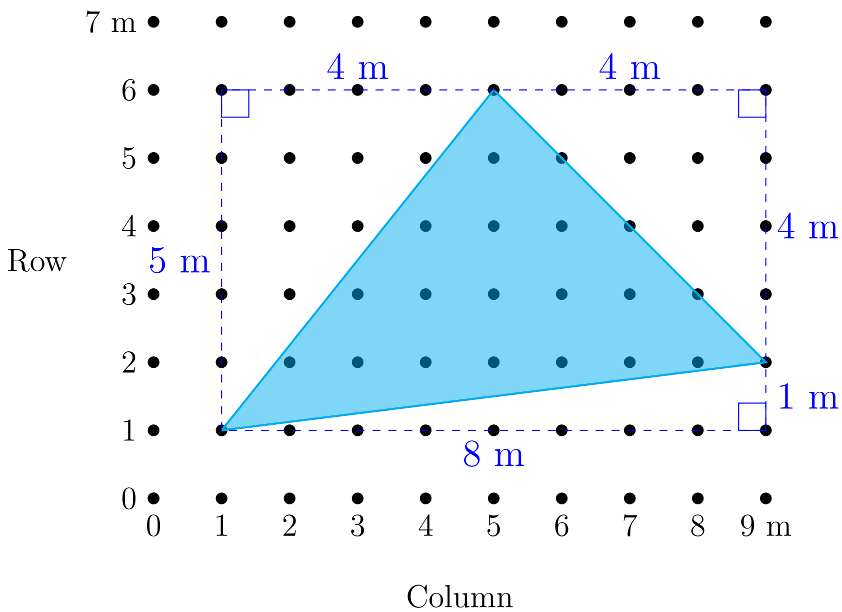
Complete the Rectangle

If we have a triangle, $\triangle ABC$, and we enclose it in a rectangle, then

$$\mathrm{A}_{\triangle ABC} = \mathrm{A}_{rectangle} - \mathrm{A}_{①} - \mathrm{A}_{②} - \mathrm{A}_{③}$$

where ①, ②, and ③ are triangles that form a rectangle with $\triangle ABC$.

Example 4: Alain has a 7 m × 9 m backyard and decides to construct a triangular-shaped patio. He plots his backyard plan on a graph as follows:



What is the area of his patio? How much space does he have left in his backyard?

$$\begin{aligned}
\mathrm{A}_{patio} &= \mathrm{A}_{rectangle} - \mathrm{A}_{①} - \mathrm{A}_{②} - \mathrm{A}_{③} \\
&= (5 \times 8) - \frac{5 \times 4}{2} - \frac{4 \times 4}{2} - \frac{8 \times 1}{2} \\
&= 40 - 10 - 8 - 4 \\
&= 18 \text{ m}^2
\end{aligned}$$

The area of Alain's patio is 18 m^2.

After building his patio, he will have $(7 \times 9) - 18 = 63 - 18 = 45$ m^2 of his backyard left.

Method 4: Shoelace Theorem

Also known as "Shoelace Formula," or "Gauss' Area Formula"

> **Shoelace Theorem** (for a Triangle)
>
> Suppose a triangle has the following coordinates: (a_1, b_1), (a_2, b_2), (a_3, b_3) where a_1, a_2, a_3, b_1, b_2, and b_3 can be any positive number. Then,
>
> $$\text{A}_3 = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}\left|(a_2b_1 + a_3b_2 + a_1b_3) - (a_1b_2 + a_2b_3 + a_3b_1)\right|$$
>
> where $|a|$ is called the *absolute value* of a. (i.e. $|-3| = 3$, $|4| = 4$)

Example 5: Given the coordinates below, what is the area of $\triangle ABC$?

$$A(3,4) \qquad B(7,9) \qquad C(11,4)$$

$$\begin{aligned}
\text{A}_3 &= \frac{1}{2}\begin{vmatrix} 3 & 4 \\ 7 & 9 \\ 11 & 4 \\ 3 & 4 \end{vmatrix} \\
&= \frac{1}{2}\left|(7 \times 4 + 11 \times 9 + 3 \times 4) - (3 \times 9 + 7 \times 4 + 11 \times 4)\right| \\
&= \frac{1}{2}\left|(28 + 99 + 12) - (27 + 28 + 44)\right| \\
&= \frac{1}{2}\left|139 - 99\right| \\
&= \frac{1}{2}\left|40\right| \\
&= \frac{1}{2} \times 40 \\
\text{A}_3 &= 20 \text{ m}^2
\end{aligned}$$

The area of $\triangle ABC$ is 20 units^2.

We can actually generalize the Shoelace Theorem and use it to find the area of a shape with any number of sides!

Shoelace Theorem (for a shape with n sides)

Suppose a shape has n sides, so there are n sets of coordinates: $(a_1, b_1), (a_2, b_2), ..., (a_n, b_n)$. Then,

$$A_n = \frac{1}{2}\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \\ \vdots & \vdots \\ a_n & b_n \\ a_1 & b_1 \end{vmatrix} = \frac{1}{2}\left|(a_2b_1 + a_3b_2 + \ldots + a_1b_n) - (a_1b_2 + a_2b_3 + \ldots + a_nb_1)\right|$$

Example 6: Calculate the area of a hexagon with the following coordinates:

(3,9), (9,9), (11,5), (9,1), (3,1), (1,5)

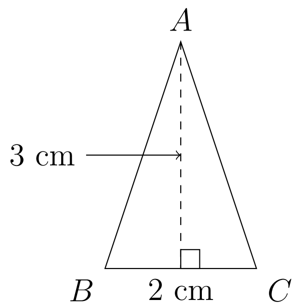
$$\begin{aligned} A_6 &= \frac{1}{2}\begin{vmatrix} 3 & 9 \\ 9 & 9 \\ 11 & 5 \\ 9 & 1 \\ 3 & 1 \\ 1 & 5 \\ 3 & 9 \end{vmatrix} \\ &= \frac{1}{2}|(9 \times 9 + 11 \times 9 + 9 \times 5 + 3 \times 1 + 1 \times 1 + 3 \times 5) \\ &\quad -(3 \times 9 + 9 \times 5 + 11 \times 1 + 9 \times 1 + 3 \times 5 + 1 \times 9)| \\ &= \frac{1}{2}\left|(81 + 99 + 45 + 3 + 1 + 15) - (27 + 45 + 11 + 9 + 15 + 9)\right| \\ &= \frac{1}{2}\left|244 - 116\right| \\ &= \frac{1}{2}\left|128\right| \\ &= \frac{1}{2} \times 128 \\ A_6 &= 64 \text{ units}^2 \end{aligned}$$

The area of the hexagon is 64 units^2.

Problem Set

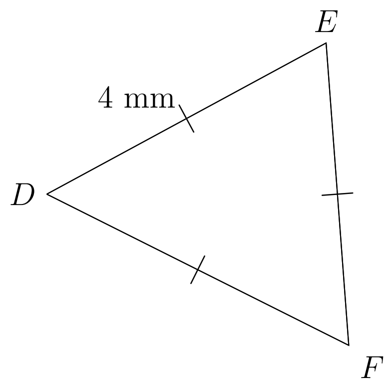
1. Find the area of the following triangles:

   (a)



$$A_{\triangle ABC} = \frac{2 \times 3}{2} = 3 \text{ cm}^2$$

   (b)



Use Heron's Formula:

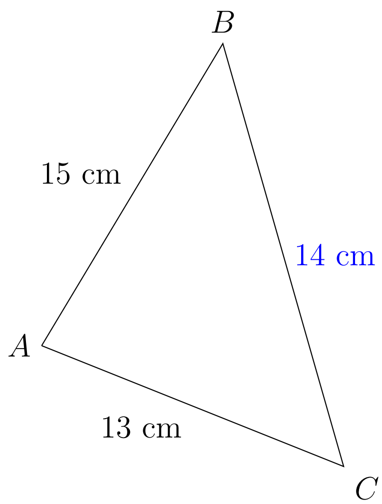
$$s = \frac{1}{2}(4 + 4 + 4) = 6$$

$$\mathrm{A}_{\triangle DEF} = \sqrt{6(2)(2)(2)} = \sqrt{48} \approx 6.928 \text{ mm}^2$$

   (c) $\triangle DEF$ where $D(2, 8), E(8, 0), F(4, 4)$

$$\begin{aligned}
\mathrm{A}_{\triangle DEF} &= \frac{1}{2}\begin{vmatrix} 2 & 8 \\ 8 & 0 \\ 4 & 4 \\ 2 & 8 \end{vmatrix} = \frac{1}{2}\left|(8 \times 8 + 4 \times 0 + 2 \times 4) - (2 \times 0 + 8 \times 4 + 4 \times 8)\right| \\
&= \frac{1}{2}\left|72 - 64\right| \\
&= \frac{1}{2}\left|8\right| \\
\mathrm{A}_{\triangle DEF} &= 4 \text{ units}^2
\end{aligned}$$

2. Given the perimeter, $\mathrm{P}_{\triangle ABC} = 42$ cm, what is the area of $\triangle ABC$?



First, we need to find x using the perimeter.

$$\text{P}_{\triangle ABC} = 42 = x + 15 + 13 \Rightarrow x = 14 \text{ cm}$$

Now, we can use Heron's formula to find the area.

$$s = \frac{1}{2}(14 + 15 + 13) = 21 \text{ cm}$$
$$\text{A}_{\triangle ABC} = \sqrt{21(21-14)(21-15)(21-13)} = \sqrt{21(7)(6)(8)} = \sqrt{7056} = 84 \text{ cm}^2$$

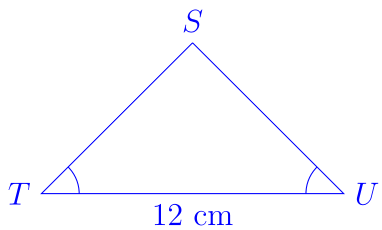
3. Given $DE = 12$, $EF = 16$, and $FD = 20$, what is the area of $\triangle DEF$?

Using Heron's Formula, we get:

$$s = \frac{1}{2}(12 + 16 + 20) = 24$$
$$\text{A}_{\triangle DEF} = \sqrt{24(24-12)(24-16)(24-20)} = \sqrt{24(12)(8)(4)} = \sqrt{9216} = 96 \text{ units}^2$$

4. The perimeter of $\triangle STU$ is 32 cm. If $\angle STU = \angle SUV$ and $TU = 12$ cm, what is the area of $\triangle STU$?

   Drawing $\triangle STU$ gives us:



   Since $\angle STU = \angle SUV$, $\triangle STU$ is an isosceles triangle which means that $ST = US$. Use the perimeter to solve for ST and US.

$$\mathrm{P}_{\triangle STU} = 32 = 12 + ST + US = 12 + 2ST \Rightarrow ST = \frac{32-12}{2} = 10 \text{ cm}$$

   Using Heron's Formula, we can find the area of $\triangle STU$:

$$s = \frac{1}{2}(12 + 10 + 10) = 16$$

$$\mathrm{A}_{\triangle STU} = \sqrt{16(16-12)(16-10)(16-10)} = \sqrt{16(4)(6)(6)} = \sqrt{2304} = 48 \text{ cm}^2$$

5. Find the area of a triangle with the following coordinates:

   (2,3), (6,3), (8,7)

   Using Shoelace Theorem, we get that...

$$\begin{aligned}
\mathrm{A}_3 &= \frac{1}{2}\begin{vmatrix} 2 & 3 \\ 6 & 3 \\ 8 & 7 \\ 2 & 3 \end{vmatrix} \\
&= \frac{1}{2}\left|(6\times 3 + 8\times 3 + 2\times 7) - (2\times 3 + 6\times 7 + 8\times 3)\right| \\
&= \frac{1}{2}\left|56 - 72\right| \\
&= \frac{1}{2}\left|-16\right| \\
&= \frac{1}{2}(16) \\
\mathrm{A}_3 &= 8 \text{ units}^2
\end{aligned}$$

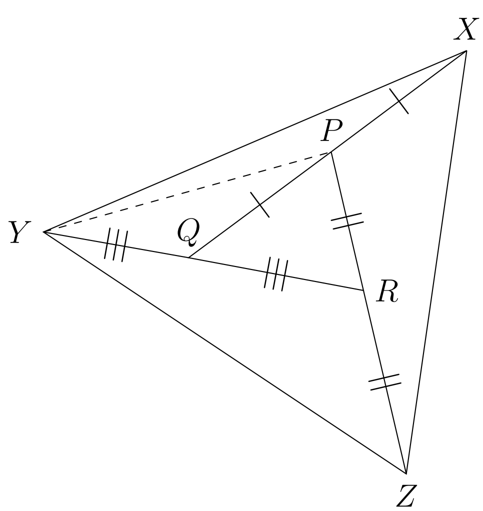
6. Locations on the main floor of a house are described using coordinate geometry. Vroom is a robotic vacuum that moves around the main floor of a house. The robot travels from one point to the next point in one straight line. Vroom's base is located at (1,1). Vroom starts at its base and moves to the following points, in order, before returning to its base: (24,10), (20,15), (12,16). What is the area of the figure that Vroom has traced out?

   Use the Shoelace Theorem!

$$\begin{aligned} A_4 &= \frac{1}{2}\begin{vmatrix} 1 & 1 \\ 24 & 10 \\ 20 & 15 \\ 12 & 16 \\ 1 & 1 \end{vmatrix} \\ &= \frac{1}{2}\left|(24\times 1 + 20\times 10 + 12\times 15 + 1\times 16) - (1\times 10 + 24\times 15 + 20\times 16 + 12\times 1)\right| \\ &= \frac{1}{2}\left|420 - 702\right| \\ &= \frac{1}{2}\left|-282\right| \\ &= \frac{1}{2}(282) \\ A_4 &= 141 \text{ units}^2 \end{aligned}$$

   The area of the region traced out by Vroom is 141 units^2.

7. $\triangle PQR$ has side QP extended to X so that $QP = PX$, PR extended to Z so that $PR = RZ$, and RQ extended to Y so that $RQ = QY$. If the area of $\triangle XYZ$ is 420 units^2, calculate the area of $\triangle PQR$. Image is not to scale. (*Problems, Problems, Problems, Volume 5: page 30, question 12*)



   Let the area of $\triangle PQR$ be t. Join YP. The areas of $\triangle YQP$ and $\triangle PQR$ are equal since they have the same height and equal bases. Similarly, the areas of triangles YQP and YPX are equal. Then the area of $\triangle YQX$ is $2t$. Similarly, the areas of triangles XPZ and YRZ are each $2t$. Therefore, $t + 2t + 2t + 2t = 420$ which gives us that $t = 60$

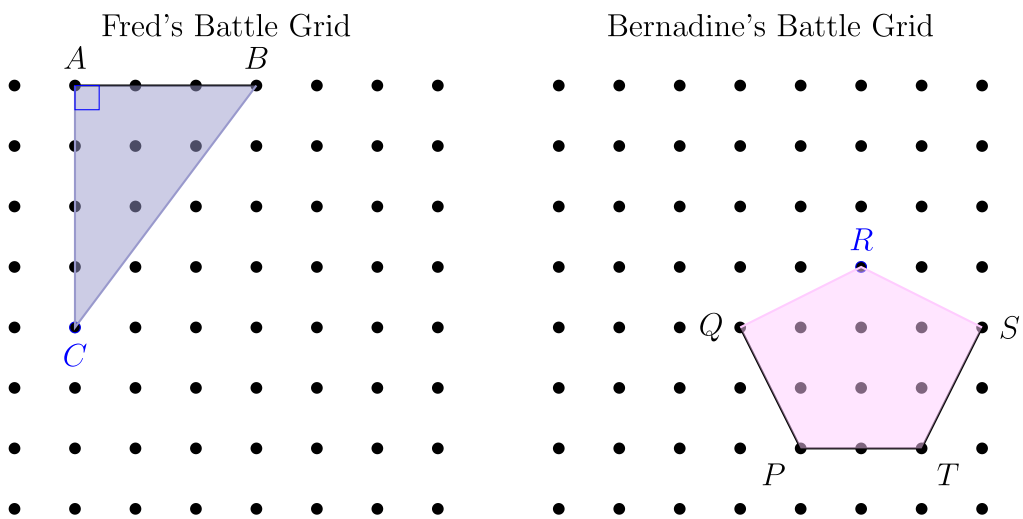
   The area of $\triangle PQR$ is 60 units^2.

8. * Bernadine and Fred are playing *Battleship: Geometry Edition* where ships sink with one hit. Fred has a triangle-shaped ship with the following coordinates:

$$A(1,7),\ B(4,7),\ C(1,x)$$

and Bernadine has a pentagon-shaped ship with the following coordinates:

$$P(4,1),\ Q(3,3),\ R(5,y),\ S(7,3),\ T(6,1)$$



(a) Fred's ship is in the shape of a right-angled triangle. If the area of Fred's ship is 6 units^2, what is the value of x? Draw the rest of Fred's ship on his grid.

Since we know the area is 6 units^2 and the triangle is a right-angled triangle, we can use the basic formula for area to find the height, $h(= BC)$. We can say AB $(= 3)$ units is the base of our triangle (i.e. $b = AB$).

$$\begin{aligned} \mathrm{A}_{\triangle ABC} = 6 &= \frac{3 \times h}{2} \\ 2\times\ 6 &= \frac{3 \times h}{\cancel{2}} \times \cancel{2} \\ \frac{12}{3} &= \frac{\cancel{3} \times h}{\cancel{3}} \\ 4 \text{ units} &= h \end{aligned}$$

Since the height of Fred's triangle-shaped ship is 4 units, $x = 7 - 4 = 3$. Thus, $C(1, x) = C(1,3)$.

(b) Bernadine's ship is in the shape of a pentagon. If the area of Bernadine's ship is 8 units^2, what is the value of y? Draw the rest of Bernadine's ship on her grid.

Since we know the area of the pentagon-shaped ship is 8 units^2 and we know all coordinates except for R, we can use the Shoelace Theorem and solve for y.

$$\begin{aligned}
\mathrm{A}_{PQRST} &= \frac{1}{2}\begin{vmatrix} 4 & 1 \\ 3 & 3 \\ 5 & y \\ 7 & 3 \\ 6 & 1 \\ 4 & 1 \end{vmatrix} \\
8 &= \frac{1}{2}\big|(3\times 1+5\times 3+7\times y+6\times 3+4\times 1) \\
&\qquad -(4\times 3+3\times y+5\times 3+7\times 1+6\times 1)\big| \\
8 &= \frac{1}{2}\left|(7y+\cancel{40})-(3y+\cancel{40})\right| \\
8 &= \frac{1}{2}\left|4y\right| \\
2\times\ 8 &= \frac{4y}{\cancel{2}}\times\cancel{2} \\
\frac{16}{4} &= \frac{\cancel{4}y}{\cancel{4}} \\
4 &= y
\end{aligned}$$

Therefore, $y = 4$.

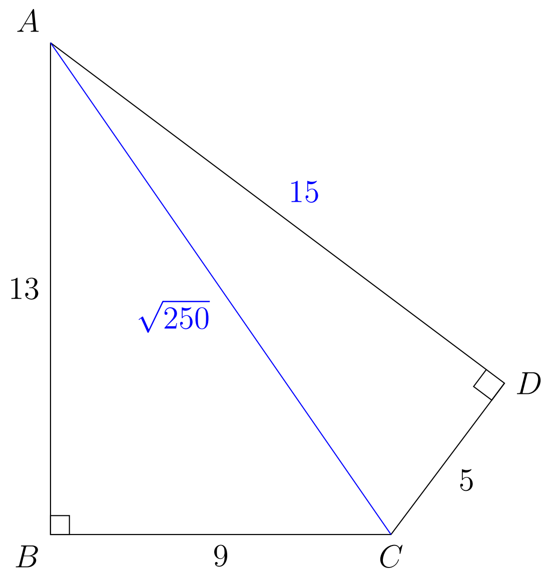
(c) If Fred attacked Bernadine at (4,3), does Fred sink Bernadine's ship?

No, he does not. Unfortunately, Fred has poor aim.

(d) If Bernadine attacked Fred at (2,6), does Bernadine sink Fred's ship?

Yes, Fred's ship sinks.

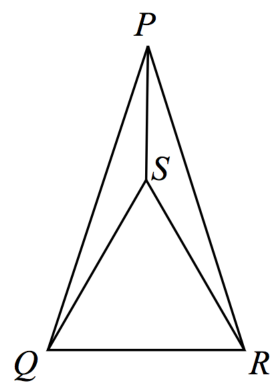
9. * In the diagram, what is the area of quadrilateral $ABCD$?



Draw line segment AC to create two right-angled triangles, $\triangle ABC$ and $\triangle ACD$. By Pythagorean's Theorem, $AC = \sqrt{9^2 + 13^2} = \sqrt{250} \approx 15.81$. Again, by Pythagorean's Theorem, $AD = \sqrt{AC^2 - 5^2} = \sqrt{250 - 25} = \sqrt{225} = 15$. The area of $ABCD$ is the sum of the areas of $\triangle ABC$ and $\triangle ACD$. Both triangles are right-angled, so we can use the basic formula to find the area of both triangles.

$$\text{A}_{ABCD} = \text{A}_{\triangle ABC} + \text{A}_{\triangle ACD} = \frac{9 \times 13}{2} + \frac{5 \times 15}{2} = \frac{117 + 75}{2} = \frac{192}{2} = 96 \text{ units}^2$$

10. ** In the diagram, $\triangle PQR$ is isosceles with $PQ = PR = 39$ and $\triangle SQR$ is equilateral with side length 30. What is the area of $\triangle PQS$? (Round your answer to the nearest whole number.)



To find the area of $\triangle PQS$, we will calculate the area of $PQSR$ by subtracting the area of $\triangle SQR$ from $\triangle PQR$. Use Heron's formula to calculate the area of both triangles.

$$s_{\triangle PQR} = \frac{1}{2}(39 + 39 + 30) = 54$$

$$\mathrm{A}_{\triangle PQR} = \sqrt{54(54-39)(54-39)(54-30)} = \sqrt{54(15)(15)(24)} = \sqrt{291600} = 540$$

$$s_{\triangle SQR} = \frac{1}{2}(30 + 30 + 30) = 45$$

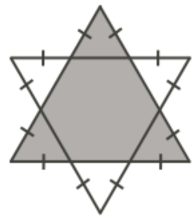
$$\mathrm{A}_{\triangle SQR} = \sqrt{45(45-30)(45-30)(45-30)} = \sqrt{45(15)(15)(15)} = \sqrt{3375} \approx 389.71$$

Now, $\mathrm{A}_{PQSR} = \mathrm{A}_{\triangle PQR} - \mathrm{A}_{\triangle SQR} = 540 - 389.71 = 150.29$. The area of $\triangle PQS$ is half of the area of $PQSR$ since $PQ = PR$, $SQ = SR$, and $\triangle PQS$ shares side length PS with $\triangle PRS$.

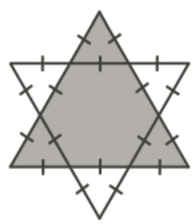
$$\mathrm{A}_{\triangle PQS} = \frac{1}{2} \times \mathrm{A}_{PQSR} = \frac{1}{2} \times 150.29 \approx 75.145 = 75$$

Therefore, the area of $\triangle PQS$ is 75 units^2.

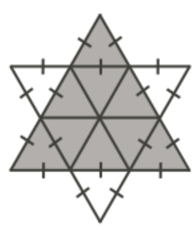
11. *** A star is made by overlapping two identical equilateral triangles, as shown below. The entire star has an area of 36 units^2. What is the area of the shaded region?



Since the triangles are equilateral, each angle of both triangles is $60°$. Each smaller triangle has two equal side lengths, marked by the single tick, so all 6 of the smaller triangles are identical isosceles triangles. Since the smaller triangles are isosceles, the missing angles can be calculated as follows: $\frac{1}{2}(180° - 60°) = 60°$. Because each angle of the small triangle is $60°$, all the small triangles are identical equilateral triangles as shown below.



Now, all side lengths of the inner hexagon are equal, so it is a regular hexagon. This mean each angle is equal to $120°$. Since a regular hexagon is symmetrical, drawing the 3 diagonals of the hexagon creates 6 more small triangles as shown below.



As it turns out, the small triangles in the inner hexagon are also identical equilateral triangles. (The diagonals split the angles of the hexagon in half, $120° \div 2 = 60°$. $\Rightarrow$ The third angle in the small triangles also equal $60°$.) The star is made of 12 small identical equilateral triangles so the area of each small triangle is $36 \div 12 = 3$. The shaded region is made of 9 small triangles and so the area of the shaded region is $9 \times 3 = 27$ units^2.