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Grade 6 Math Circles

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Matrices

Today we will be learning about a very useful mathematical tool called a matrix. A **matrix** is a rectangular arrangement of numbers. The main purpose of a matrix is to represent and organize information without labels. In this lesson we will learn how to set up a matrix or several matrices to represent a problem. We will also learn how to apply some operations that you already know to matrices in a way that you haven't seen before.

For example, suppose you and a friend each have a box of Smarties and you each want to count all your red and blue Smarties. We can represent this information in a table and then convert it into a matrix as shown below.

	Red	Blue
You	5	4
Friend	3	6

$$\Longrightarrow \begin{bmatrix} 5 & 4 \\ 3 & 6 \end{bmatrix}$$

Example 1: Suppose Garden A has 4 cabbages and 11 potatoes and Garden B has 10 cabbages and 7 potatoes. Represent this information in a matrix.

Some terms to understand

The **dimensions** of matrix are how many rows and columns it has. So far we have only seen 2×2 matrices. In general, an $m \times n$ matrix has m rows and n columns. Matrix A (shown below) is a 2×3 matrix since there are two rows and three columns.

$$A = \underbrace{\begin{bmatrix} 5 & 1 & 6 \\ 2 & 4 & 3 \end{bmatrix}}_{\text{3 columns}} \Bigg\} \text{ 2 rows}$$

If a matrix has m rows and m columns, it is an $m \times m$ matrix or a **square matrix**. The matrix below is an example of a 2×2 matrix which is a square matrix.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Example 2: What are the dimensions of the following matrices?

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 8 \\ 4 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 4 & 5 \\ 4 & 2 & 3 \\ 7 & 2 & 1 \end{bmatrix}$

The numbers inside a matrix are called **elements**. To generalize a matrix, we can label each element in our matrices using variables. For example,

2×2	3×2	2×3	3×3	3×1
$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$	$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$	$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$	$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

To refer to a certain element in a matrix we use indexing. To talk about the element in the second row and third column in matrix A we would say $a_{2,3}$. For example, in the matrix below, $a_{3,1}$ refers to 6 because it is the element in the third row and the first column.

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 4 \\ 6 & 2 \end{bmatrix}$$

Matrices in Real-life

Matrices are used in many different fields and professions. Here are a few examples:

- *Cryptography* - Matrices can be used to encrypt and decrypt secret messages. The sender encrypts the message using an encoding matrix and the receiver decrypts it using a decoding matrix.

$$\begin{array}{cccc} & \text{encoding matrix} & \text{message} & \text{encrypted message} \\ Sender \Longrightarrow & \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix} & \begin{bmatrix} 6 & 3 \\ 1 & 7 \end{bmatrix} = & \begin{bmatrix} 33 & 36 \\ 20 & 23 \end{bmatrix} \end{array}$$

$$\begin{array}{cccc} & \text{decoding matrix} & \text{encrypted message} & \text{message} \\ Receiver \Longrightarrow & \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} & \begin{bmatrix} 33 & 36 \\ 20 & 23 \end{bmatrix} = & \begin{bmatrix} 6 & 3 \\ 1 & 7 \end{bmatrix} \end{array}$$

- *Economics* - Businesses use matrices to study stock market trends to earn a larger profit and to lower their losses

$$\begin{array}{cccccc} & Revenue & & Costs & & Profit \\ November & \begin{bmatrix} 125 & 55 & 240 \\ 125 & 80 & 200 \end{bmatrix} & - & \begin{bmatrix} 50 & 75 & 60 \\ 40 & 70 & 60 \end{bmatrix} & = & \begin{bmatrix} 75 & -20 & 180 \\ 85 & 10 & 140 \end{bmatrix} \end{array}$$

- *Google* - Ever wonder how Google always knows what you search for? Or wonder how they determine what sites to direct you to? Google uses a page ranking method that uses matrices!

Matrix Operations

Since matrices usually have more than one element, matrix operations do not follow the same rules that operations between two numbers do. They have their own way of performing the usual operations of addition, subtraction, and multiplication.

Matrix Addition

To perform matrix addition, we add together elements that are in the same position in each matrix. Suppose we want to add two 2×2 matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

When we add two matrices together, do they need to have the same dimensions?
Yes because we are adding elements which have the same index (are located in the same spot in the matrices).

Example 3: Add the following matrices.

$$\begin{bmatrix} 11 & 3 & 5 \\ -1 & 8 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 8 & 6 \\ 5 & 10 & 2 \end{bmatrix} =$$

Example 4: In the 2018 Winter Olympics, Canada won 11 gold medals, 8 silver medals, and 10 bronze medals. Norway won 14 gold, 14 silver, and 11 bronze. China won 1 gold medal, 6 silver medals, and 2 bronze medals.

In the 2016 Summer Olympics, Canada won 4 gold medals, 3 silver medals, and 15 bronze. Norway won 0 gold, 0 silver, and 4 bronze while China won 26 gold, 18 silver, and 26 bronze. How many of each medal does each country have from the past two olympics?

Matrix Subtraction

Matrix subtraction works the same as addition in that we subtract elements that are in the same position in each matrix.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Like matrix addition, matrices need to have the same dimensions in order to perform matrix subtraction.

Example 5: Perform the following matrix subtraction.

$$\begin{bmatrix} 11 & 8 \\ 7 & 24 \\ 17 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 5 \\ 0 & 15 \\ 9 & 2 \end{bmatrix} =$$

Scalar Multiplication

A **scalar** is a number. To perform scalar multiplication, we multiply each element in the matrix by the same number.

$$k \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

Example 6: Calculate the following:

(a) $4 \begin{bmatrix} 1 & 10 & 3 \\ 6 & 1 & 2 \\ 5 & 7 & 4 \end{bmatrix} =$

(b) $3 \begin{bmatrix} 1 & 6 \\ 10 & 3 \end{bmatrix} =$

Example 7: Yesterday at the Elmira Maple Syrup Festival, there are were 45 kids, 12 teenagers, and 51 adults who went on the carriage ride and 31 kids, 55 teenagers, and 75 adults who went on the guided tour. Today, there were twice as many people because word got out about how fun the festival was. How many kids, teenagers, and adults went on the carriage ride today? How many took a guided tour today?

Transpose

The **transpose** is an operator which switches the row and column indices of the original matrix, A, and produces another matrix, A^T, which has rows that used to be the columns of A. For example,

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

Example 8: Determine A^T.

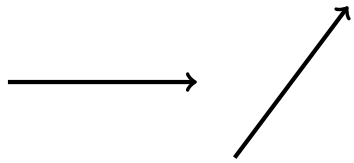
(a) $A = \begin{bmatrix} 11 & 1 \\ 5 & 14 \\ 6 & 2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 7 \\ 9 & 13 \end{bmatrix}$

(c) $A = \begin{bmatrix} 6 & 8 & 5 & 1 \\ 3 & 10 & 9 & 8 \\ 7 & 2 & 16 & 4 \end{bmatrix}$

Applications of Matrices

Recall: Vectors



As you might remember, vectors are represented with arrows. We can add two vectors together by placing them side-by-side and following the arrows.

The above diagram demonstrates the triangle method of adding vectors. Vectors can also be added using the parallelogram method which involves placing the arrows so they start at the same point.

The Area of a Parallelogram

Matrices can be used to represent vectors. For example, if we were to represent a vector that went right 4 units and up 3 units, that is, a vector that pointed at the point (4,3), with a matrix we would write it as

$$\vec{a} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

If we are given two vectors that represent two sides of a parallelogram we can calculate the area of the parallelogram (assuming one of the corners of the parallelogram is at (0,0)). We do this by taking the absolute value of the **determinant** of the 2×2 matrix formed by putting the two vectors together. The absolute value of a number is the number but *positive*. For example, the absolute value of 6 is 6 and the absolute value of -7 is 7.

First, let's look how to calculate the determinant of a 2×2 matrix.

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

Now we can find the area of a parallelogram using the below formula

$$\text{Area} = \left| \det \begin{bmatrix} \vec{p} & \vec{q} \end{bmatrix} \right|$$

Example 9: Find the area of the parallelograms using the points given.

(a) $\vec{p} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ & $\vec{q} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

(b) $\vec{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$

Problem Set

1. Construct a matrix for the following scenarios:

   (a) A class vote was conducted on which one is better: books or movies. For books, there were 10 votes from the Grades 7s and 6 votes from the Grade 8s. For movies, there were 8 votes from the Grade 7s and 11 votes from the Grade 8s.

   (b) It's the season for apple ciders and hot chocolate drinks! On Friday, Tim Horton's sells 25 apple ciders, 32 hot chocolates, and 28 white hot chocolates. On Saturday, they sell 53 apple ciders, 55 hot chocolates, and 46 white hot chocolates. On Sunday, they sell 45 apple ciders, 45, hot chocolates, and 47 white hot chocolates.

2. Determine whether the following operations are possible. If they are, state the dimensions of the produced matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad C = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \qquad D = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad E = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

   (a) $A - C$
   (b) $A + A$
   (c) $2D$
   (d) $B - C^T$
   (e) $-4E + D$

3. Add or subtract the following matrices:

$$A = \begin{bmatrix} 10 & 5 & 6 \\ 1 & 17 & 20 \end{bmatrix} \qquad B = \begin{bmatrix} 9 & 14 & 37 \\ 51 & 8 & 23 \end{bmatrix} \qquad C = \begin{bmatrix} 11 & 2 & 7 \\ 18 & 15 & 3 \end{bmatrix}$$

   (a) $A + B$
   (b) $B - C$
   (c) $A + B - C$
   (d) $B + C - A$

4. Calculate the following given matrices G and H:

$$G = \begin{bmatrix} 1 & 3 \\ 9 & 4 \end{bmatrix} \qquad H = \begin{bmatrix} 0 & 3 \\ 2 & 5 \end{bmatrix}$$

   (a) $3G + H$ (b) $5G - 4H$

5. An electronics company called Alpha wants to determine which of their phone products made the most profit (or made the most money). In the last month,

   - Phone A earned \$150, 000 and Phone B earned \$200, 000 in total
   - Phone A cost \$100, 000 and Phone B cost \$175, 000 to manufacture

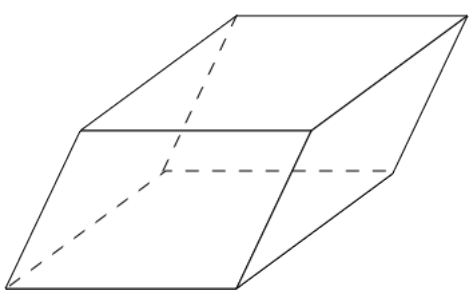
   (**Hint**: The equation to calculate profit is $Revenue - Cost = Profit$ where revenue is the amount of money earned, cost is the amount of money spent, and profit is the amount of money leftover.)

   (a) How much profit did Alpha make in the last month total?

   (b) Which phone product made the most profit in the last month?

6. * In the lesson, we found the area of a parallelogram by using a formula involving the absolute value of the determinant of a 2×2 matrix. Similarly, we can determine the volume (the space inside) of a parallelepiped using three vectors that we are given and the formula of

$$\text{Volume} = \left| \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right|$$



The determinant of a 3×3 matrix is as follows

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Given vectors $\vec{u} = \begin{bmatrix} 6 \\ 3 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$, determine the volume of the parallelepiped.

7. ** Monique, Akiko, and Martin are playing Monopoly. They each start off with only \$1500. In the beginning they all have zero properties and zero houses.

In the first trip around the board:

- Monique spends \$750 and gets 2 properties
- Akiko buys one property for \$600
- Martin buys 3 properties for \$1000 but he lands on Akiko's property and has to pay her \$20

In the second trip around the board:

- They each get \$200 for passing GO
- Monique buys another property for \$125. She also buys 3 houses for \$600
- Akiko buys one property for \$450
- Martin also makes a deal with Akiko where he gives her \$500 in exchange for one of her properties. He also buys 1 house for \$75

(a) How much money does each player have after the first trip around the board? How many properties and houses does each player have after the first trip around the board? Using matrices and matrix operations to find the answer.

(b) After the second trip around the board how much money and how many properties and houses does each player have? Use matrices and matrix operations and your answer from part (a).

8. *** *Matrix Multiplication*

Matrix multiplication involves multiplying two matrices together. To perform matrix multiplication, we multiply each element of a row from one matrix by the corresponding element of the column from the other matrix and add the product together. For a way to visualize matrix multiplication, watch this youtube video at `https://youtu.be/bFeM4ICRtOM`. For example, matrix multiplication between two 2×2 matrices looks like:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Calculate (a) and (b).

(a) $\begin{bmatrix} 4 & 6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 8 & 1 \\ 7 & 2 \end{bmatrix} =$

(b) $\begin{bmatrix} 7 & 1 \\ 2 & 5 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 5 & 3 & 4 \end{bmatrix} =$

(c) When you multiply two matrices together is the result always a square matrix? **Hint:** What is the requirement that A and B must meet in order to multiply A and B together like $A \times B$?