Non-Euclidean Geometry and the Globe

(Euclidean) Geometry Review:

What is Geometry?

Geometry asks the question: what can we know about shapes?

2200 years ago, Euclid decided to answer that question, and wrote *Elements*. Almost every bit of geometry you will ever see done will have come from that book.

In *Elements*, Euclid made the rules for geometry using only two things:

1. Compass: The ability to make a circle by choosing a point as a center, and/or a second point to set the radius of the circle.

2. Straight edge: The ability to make a straight line using two points.

This is all geometry done on a flat piece of paper.
Euclidean Geometry Examples
With just a compass and straight edge, you can make:

Polygons:

Circles:
Equilateral Triangles:

Regular Polygons:
Parallel Lines
What are parallel lines? In *Elements*, Euclid defines parallel lines with his parallel postulate:

*If two lines are intersected by a third line such that the sum of interior angles formed on one side is less than the sum of two right angles, then the original two lines will intersect on that side (if extended indefinitely).*

*If the interior angles formed sum to two right angles on both sides, then the original two lines will never intersect. These two lines in this case would be parallel.*

\[ 60° + 120° = 180° = \text{two right angles} \]

*Note: Try for yourself. The sum of interior angles for parallel lines is always 180°*

Over time people have restated this in many ways. Another way of saying this was given by Playfair:

**IF** you have a point and a line,

**THEN** you can draw one and only one line going through the point that will never touch the original line. These two lines would be parallel

*Note: Both these definitions are talking about *straight lines* in particular.*

This way of understanding parallel lines, called Playfair’s axiom, means the exact same thing as Euclid’s parallel postulate. We will use Playfair’s axiom later in the lesson.
Parallel Lines Examples

Are these red lines parallel?

Are these red lines parallel on this map?
What if these same lines were on a different map?

What if I put these same lines on a globe?
We’re going to talk about globes as if they are spheres. Spheres are perfectly round 3D balls.

Look closely at the grid patterns on the two maps and the picture of the globe above. The red lines on these maps are exactly the same lines - it’s just the maps that look different. Using Playfair’s axiom, these lines are obviously not parallel since you can see where they touch at the North and South poles on the second map and on the globe.

As a matter of fact, we will see that Euclid’s parallel postulate (and so also Playfair’s axiom) is not even true on a globe. Notice that on the picture of the globe, the equator is perpendicular to both of the red lines. This means that the “sum of the interior angles” where the red lines are intersected by the equator is equal to two right angles (180°), but these lines are still not parallel. This tells us that the regular geometry we already know doesn’t work if we’re drawing on a globe. We need to do something different.

*Note: The above picture has an extra vertical line so you can see the interior angles better.
Non-Euclidean Geometry

You can think of non-Euclidean geometry as geometry that can’t be done with a compass and straight edge in the way that Euclid would do it. In our case, we’re talking about geometry on surfaces that are not flat (in particular spherical geometry on a globe).

Exercise: Triangles and Interior Angles

Let’s review what a triangle is, and some of its properties:

- A triangle is three straight lines, coming together to make three vertices.
- The sum of interior angles of a triangle is 180°.
- The main types of triangles are: Equilateral, Isoceles, and Right Angle.

So, let’s try drawing a triangle on the globe:

1. Start by drawing a line from the equator to the North pole of your globe.
2. From the North pole, 90° to draw a line that goes back to a different spot on the equator.
3. From here, draw a line along the equator that connects the lines you just drew.

You now in fact have an equilateral triangle.

What is the sum of the interior angles here? If you look at each of the vertices closely, you’ll notice that you have three 90° angles. This means the sum of the interior angles of the triangle you just drew is 270°! This shows us again that geometry on a globe must be different from what it’s like on flat paper.
**Straight Lines and Curved Surfaces**

When we talked about straight lines and triangles in Euclidean geometry, we always said that we were using straight lines. Are the lines we’ve drawn on the globe straight? What does it mean for a line to be straight on the globe?

Think about this: what makes a straight line straight? What do straight lines do on a flat surface? Think back to vectors (covered last lesson): A straight line between two points on a flat surface is the shortest way to get from one point to the other. So on a curved surface, a straight line should do the same thing.

This is actually an important part of non-Euclidean geometry: what do you call a straight line in your space? From this we know what parallel lines and shapes are like in the space. The line that is the shortest distance between two points on a surface is called a **geodesic**.

**Straight Lines on a Sphere**

What do you think would be the shortest path to get from one point to another on a sphere? In other words: what is the **geodesic** on a sphere? It’s actually a segment of a **great circle**: the biggest circle you can draw that goes all the way around the sphere.

On the globe, the Equator is a great circle. Any line of longitude is also a great circle. The lines of latitude are NOT great circles, other than the equator. These are not the only great circles on a sphere, but they’re the easiest ones to see on globe. This also shows us how Playfair’s axiom is also false on a globe: ANY two “straight lines” on a globe will ALWAYS intersect.
Maps and Projections

Earlier in the lesson we talked about how the same lines on a globe can look different depending on the map you use to look at them. Why do you think these maps look so different? We know that the actual Earth is similar to a sphere, which is why we have globes. But a map is never a sphere. Maps are always drawn on flat pieces of paper. To be able to do this, mathematicians use projections.

The map above, which is one of the most common maps, is made using the Mercator projection. The idea is to put the globe inside a cylinder, and make points on the cylinder according to a specific equation. Once you unroll this cylinder, you have a map!

What this means for us though is that we can’t just draw lines on a map that look straight and call them parallel. Although a map is flat, any map is really just a projection of the globe. So the geometry that really matters in this case is the geometry that we can do on the globe.

Maps are also often made with a goal in mind. The Mercator projection was made to keep a 90° angle between lines of longitude and latitude. This was very helpful to navigators across the ocean, because it meant they could leave at a certain angle, and get to where they wanted to go without having to recalculate the angle.
In Summary

In the big picture, we learned:

- Euclidean geometry includes most geometry you’ll see done in a flat space (and all you need is a compass and straight edge).

- Euclid’s parallel postulate defines what parallel lines are on a flat surface. Playfair’s axiom does the same thing, and is a useful way to understand parallel lines.

- Non-Euclidean geometry is just geometry that is not Euclidean. In particular: We looked at geometry on surfaces that are not flat.

We looked more deeply at spherical geometry today:

- Math on a sphere (or any non-flat surface) can be very different from math on a flat piece of paper.
  - The properties we’re used to shapes and lines having aren’t always true on surfaces that aren’t flat.

- Geodesic: Straight lines in spherical geometry are great circles.

- We can use Projections on a globe to get a map, but doing that will always change something in how you’re seeing the world.
Problems

REVIEW

1. What is the sum of the interior angles of a triangle in Euclidean geometry? What is
   the sum of the interior angles of an equilateral triangle in spherical geometry?

2. In *Elements*, what two tools does Euclid use to do geometry?

3. (a) Given a point and a straight line, how many straight lines can you draw through
   the point that are parallel to the first line on a flat piece of paper?

   (b) Given a point and a geodesic line, how many geodesic lines can you draw through
   the point that are parallel to the first geodesic on a sphere?

4. What are the sums of the interior angles of the following triangles?

   ![Triangle Diagrams](image)

APPLY

The drawings below do not need to be done to complete accuracy, since measuring angles
on a sphere can be hard.

5. If you can, go to https://www.euclidea.xyz/en/game/ to do some levels of geometry
   with only a ruler and straight edge.

6. Draw a triangle on a globe that does NOT have a line on the Equator.

7. Draw a square on the globe. The sum of interior angles in a Euclidean square is 360°.
   Is the the sum of interior angles of a square in spherical geometry bigger or smaller?
8. The Mercator projection was very useful for sailing across the ocean. The path that boats took across the ocean was a straight diagonal line on the Mercator projection. What do you think the path would look like on a globe? Would it be a straight line?

![Path on Mercator projection](image)

**THINKING**

9. Spherical geometry, which we talk about in this lesson, is a type of *elliptical geometry*. Elliptical geometry is a non-Euclidean geometry where

   **IF** you have a point and a straight line,

   **THEN** you can draw NO straight lines through the point that will never touch the original line.

In other words, any geodesics you draw will always intersect, and there is no such thing as parallel lines. Can you think of any other types of non-Euclidean geometry? Think about how else you can change rules about parallel lines. Would the sum of internal angles of a triangle be bigger, equal to, or less than 180° in this new geometry?